Conservative discretizations for the point vortex problem on the plane

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Abstract

Equations governing the interaction of vortices were investigated, and point vortex model has been used to obtain set of ODEs. General algorithm to find the conservation law multipliers and their corresponding conservation laws was applied to systems such as the harmonic oscillator, gravitational 2-body problem and in detail to 2- vortex problem. Obtained multipliers and conserved quantities from 2-vortex problem were generalized to N-vortex problem, then they were discretized for N=2, N=3. Resulting discretizations were then used to find a numerical scheme for the 3-vortex ODE that preserves the conservation laws at the discrete level. A numerical solution was obtained by coding the obtained schemes. The behavior of the conserved quantities were analyzed and compared with standard schemes.

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Chapter 1

Introduction

This section will present the derivation of point-vortex problem on a plane from incompressible Navier-Stokes equation. Crucial assumptions that lead to the point vortex equation will be stated. Furthermore, the Hamiltonian structure of the equations will be discussed.

1.1 Derivation of the vorticity equation on the plane

The dynamics of fluids is governed by Navier-Stokes equations. For in-compressible flow in 2-D Navier-Stokes equation becomes,

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v}\right) = -\nabla P + \mu\nabla^2\vec{v},$$
$$\nabla\cdot\vec{v} = 0,$$

where, $(\vec{v}\cdot\nabla)$ is the Jacobian matrix of the velocity vector field $\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j}$. If the fluid is invicid $(\mu = 0)$, viscous term $\nabla^2\vec{v}$ is no longer present which results in Euler's equations,

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v}\right) = -\nabla P,$$
$$\nabla\cdot\vec{v} = 0.$$

The vorticity(ω) of the flow is a measure of rotation of a fluid parcel and is defined as the curl of the velocity field($\nabla\times\vec{v} = \vec{w} = w\hat{k}$). Taking the curl on both sides of the Euler's equation yields,

$$\rho\left(\nabla\times\frac{\partial \vec{v}}{\partial t} + \nabla\times(\vec{v}\cdot\nabla)\,\vec{v}\right) = \nabla\times(-\nabla P) = 0. \tag{1.1}$$

Using the vector identities,

$$(\vec{v}\cdot\nabla)\,\vec{v} = \nabla\left(\frac{\vec{v}\cdot\vec{v}}{2}\right) - \vec{v}\times\vec{w},$$

$$\nabla\times(\vec{v}\cdot\nabla)\,\vec{v} = \nabla\times\nabla\left(\frac{\vec{v}\cdot\vec{v}}{2}\right) - \nabla\times(\vec{v}\times\vec{w}) = (\vec{v}\cdot\nabla)\,\vec{w} - (\vec{w}\cdot\nabla)\,\vec{v} + \vec{w}\,(\nabla\cdot\vec{v}) + \vec{v}\,(\nabla\cdot\vec{w}),$$

incompressibility, furthermore realizing that $\nabla\cdot\vec{\omega} = 0$ and $(\vec{w}\cdot\nabla)\,\vec{v} = 0$ in 2-D , equation[1.1] simplifies to,

$$\frac{D\vec{w}}{Dt} = \frac{\partial\vec{w}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{w} = 0$$

Thus the material derivative of vorticity being zero implies that the vorticity of a fluid parcel is constant in time, and vorticity is conserved.In order to yield simpler equations we will assume that vorticity field is manifested by point vortices which can be represented by Dirac delta distribution.In this assumption the vorticity is infinite at the vortex and zero everywhere else. Point vortices will create vorticity field which has a corresponding velocity field. Each point vortex will create a vorticity field which acts on all vortices except itself. Consequently each point vortex will get affected by the superposition of vorticity fields of other vortices. Resulting vorticity field due to N vortices is,

$$w(\vec{r}) = \sum_{\alpha=1}^{N} \Gamma_\alpha \delta(r - r_\alpha),$$

where Γ_α is the circulation strength of the α-th point vortex.
By the Helmholtz decomposition, any 2D vector field($\vec{v}$) which is sufficiently smooth and rapidly decaying at infinity can be resolved into a sum of solenoidal and irrotational vector fields such that,

$$\vec{v} = -\nabla\Phi + \nabla \times (\Psi\hat{k}). \tag{1.2}$$

Taking the curl of both sides of equation results in Poisson's Equation and noticing that in 2-D, $\nabla \cdot (\Phi\hat{k}) = 0$, leads to

$$\vec{w} = \nabla \times \vec{v} = \nabla \times (-\nabla\Phi) + \nabla \times \nabla \times \vec{\Psi} = \nabla\left(\nabla \cdot \vec{\Psi}\right) - \nabla^2\vec{\Psi},$$

$$\Rightarrow w = -\nabla^2\vec{\Psi}.$$

The solution is given by $\Psi(\vec{r}) = \int G(r,r')w(r')dr'$, where $G(r,r')$ is the Green's Function given by,

$$G(r,r') = -\frac{1}{4\pi}\ln\|r - r'\|^2.$$

Thus,

$$\Psi(\vec{r}) = \int -\frac{1}{4\pi}\ln\|r - r'\|^2 \left(\sum_{\alpha=1}^{N} \Gamma_\alpha \delta(r - r_\alpha)\right) dr',$$

$$\Psi(r) = -\frac{1}{2\pi}\sum_{\alpha=0}^{N} \Gamma_\alpha \ln\|r - r_\alpha\| \tag{1.3}$$

Note that incompressibility condition is satisfied since $\vec{v} = \frac{\partial\Psi}{\partial y}\hat{\imath} - \frac{\partial\Psi}{\partial x}\hat{\jmath} = -\nabla \times \Psi$ and so,

$$\nabla \cdot \vec{v} = \frac{\partial^2\Psi}{\partial y \partial x} - \frac{\partial^2\Psi}{\partial x \partial y} = 0$$

1.2 Point vortex problem on the plane

Hence by [1.2] and[1.3],the velocity vector for each point vortex is given by

$$\dot{x}_\alpha = \frac{\partial\Psi}{\partial y_\alpha}, \qquad \dot{y_\alpha} = -\frac{\partial\Psi}{\partial x_\alpha},$$

$$\rightarrow \frac{dx_\alpha}{dt} = \frac{-1}{2\pi}\sum_{\beta=1,\beta\neq\alpha}^{N} \frac{\Gamma_\beta(y_\alpha - y_\beta)}{r^2_{\alpha\beta}}, \qquad \frac{dy_\alpha}{dt} = \frac{1}{2\pi}\sum_{\beta=1,\beta\neq\alpha}^{N} \frac{\Gamma_\beta(x_\alpha - x_\beta)}{r^2_{\alpha\beta}}, \qquad for \qquad \alpha = 1,2,3.. \tag{1.4}$$

One special feature of these equations is that it is a Hamiltonian system. By re-scaling the equations one could rewrite [1.4] as ,

$$\Gamma_\alpha \frac{dx_\alpha}{dt} = \frac{\partial H}{\partial y_\alpha}, \qquad \Gamma_\alpha \frac{dy_\alpha}{dt} = -\frac{\partial H}{\partial x_\alpha}, \qquad \alpha = 1, 2, 3 \dots N,$$

where

$$H = -\frac{1}{4\pi} \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} \Gamma_\alpha \Gamma_\beta ln|r_{\alpha\beta}|.$$

Physically, the Hamiltonian represents the kinetic energy of the fluid,

$$K.E = \frac{1}{2} \int \|v\|^2 dA = \frac{1}{2} \int w\Psi dA = -\frac{1}{4\pi} \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} \Gamma_\alpha \Gamma_\beta ln|r_{\alpha\beta}|.$$

Chapter 2

Conservation Law Multipliers

In this section, the theory of conservation law multipliers will be briefly described. In particular, the general method for obtaining conservation law multipliers and associated conservation laws. In order to illustrate the method, the theory will be applied on various system of differential equations.

2.1 Theory of conservation law multipliers

We review the theory of conservation law multipliers from [2, 1] for ODEs.
Let $\vec{F}(t, \vec{x}, \dot{\vec{x}}, \ddot{\vec{x}}, \dddot{\vec{x}} \cdots) = 0$ represent a system of differential equations. For each conserved quantity $\Phi(t, \vec{x}, \dot{\vec{x}}, \ddot{\vec{x}}, \dddot{\vec{x}} \cdots)$ there exists one or more conservation law multiplier,$\vec{\lambda}(t, \vec{x}, \dot{\vec{x}}, \ddot{\vec{x}}, \dddot{\vec{x}} \cdots)$ such that,

$$\vec{\lambda} \cdot \vec{F} = D_t \left(\tilde{\Phi} \right) = 0,$$

on $\vec{F} = 0$ where $\tilde{\Phi}$ is an equivalent conservation law of Φ. This means that $\lambda \cdot \vec{F}$ must be the total derivative of some scalar function Φ with respect to t. Combining all multiplier vectors in one matrix, as row vectors, one can construct the multiplier matrix Λ . Moreover, one can represent corresponding conserved quantities as a single vector $\vec{\Phi}$, to obtain,

$$\Lambda \vec{F} = D_t \left(\vec{\Phi} \right).$$

The conservation law multipliers $\vec{\Lambda}$ can be found by the method of Euler-Lagrange Operator. If $\lambda \cdot \vec{F}$ is a total derivative then Euler-Lagrange operator $\vec{\xi()}$ applied to $\vec{\lambda} \cdot \vec{F}$ should vanish for arbitrary smooth function $x\vec{(t)}$. In other words,

$$\xi(\overrightarrow{\vec{\lambda} \cdot \vec{F}}) = \begin{pmatrix} \frac{\partial \vec{\lambda} \cdot \vec{F}}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial \vec{\lambda} \cdot \vec{F}}{\partial \dot{x_1}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial \vec{\lambda} \cdot \vec{F}}{\partial \ddot{x_1}}\right) + \cdots + (-1)^n \frac{d^n}{dt^n}\left(\frac{\partial \vec{\lambda} \cdot \vec{F}}{\partial x_1^n}\right) \\ \frac{\partial \vec{\lambda} \cdot \vec{F}}{\partial x_2} - \frac{d}{dt}\left(\frac{\partial \vec{\lambda} \cdot \vec{F}}{\partial \dot{x_2}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial \vec{\lambda} \cdot \vec{F}}{\partial \ddot{x_2}}\right) + \cdots + (-1)^n \frac{d^n}{dt^n}\left(\frac{\partial f}{\partial x_2^n}\right) \\ \frac{\partial \vec{\lambda} \cdot \vec{F}}{\partial x_3} - \frac{d}{dt}\left(\frac{\partial \vec{\lambda} \cdot \vec{F}}{\partial \dot{x_3}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial \vec{\lambda} \cdot \vec{F}}{\partial \ddot{x_3}}\right) + \cdots + (-1)^n \frac{d^n}{dt^n}\left(\frac{\partial f}{\partial x_3^n}\right) \\ . \\ . \\ . \\ \frac{\partial \vec{\lambda} \cdot \vec{F}}{\partial x_m} - \frac{d}{dt}\left(\frac{\partial \vec{\lambda} \cdot \vec{F}}{\partial \dot{x_m}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial \vec{\lambda} \cdot \vec{F}}{\partial \ddot{x_m}}\right) + \cdots + (-1)^n \frac{d^n}{dt^n}\left(\frac{\partial \vec{\lambda} \cdot \vec{F}}{\partial x_m^n}\right) \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^n (-1)^i \frac{d^i}{dt^i}\left(\frac{\partial \vec{\lambda} \cdot \vec{F}}{\partial x_1^i}\right) \\ \sum_{i=0}^n (-1)^i \frac{d^i}{dt^i}\left(\frac{\partial \vec{\lambda} \cdot \vec{F}}{\partial x_2^i}\right) \\ \sum_{i=0}^n (-1)^i \frac{d^i}{dt^i}\left(\frac{\partial \vec{\lambda} \cdot \vec{F}}{\partial x_3^i}\right) \\ . \\ . \\ . \\ \sum_{i=0}^n (-1)^i \frac{d^i}{dt^i}\left(\frac{\partial \vec{\lambda} \cdot \vec{F}}{\partial x_m^i}\right) \end{pmatrix}$$

for arbitrary smooth function $x\vec{(t)}$. We illustrate the concept by example.

2.2 Example: Harmonic Oscillator

Equation of the harmonic oscillator is given by the formula:

$$m\ddot{x} + kx = 0$$

Then, $F(x, \ddot{x}) = m\ddot{x} + kx$. Furthermore, we will assume that the multiplier is only a function of $t, x, \dot{x}$. Hence, $\lambda = \lambda(t, x, \ddot{x})$. Euler-Lagrange operator applied on arbitrary function $\vec{x(t)}$ is represented as follows:

$$\xi(\lambda F) = \frac{\partial \lambda F}{\partial x} - \frac{d}{dt}\left(\frac{\partial \lambda F}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial \lambda F}{\partial \ddot{x}}\right)$$

Evaluating $\xi(\lambda F)$:

$$\frac{\partial \lambda F}{\partial x} = \lambda_x(m\ddot{x} + kx) + \lambda k$$

$$\frac{d}{dt}\left(\frac{\partial \lambda F}{\partial \dot{x}}\right) = \lambda_{\dot{x}}(m\dddot{x} + k\dot{x}) + (m\ddot{x} + kx)(\lambda_{\dot{x}t} + \lambda_{\dot{x}x}\dot{x} + \lambda_{\dot{x}\dot{x}}\ddot{x})$$

$$\frac{d^2}{dt^2}\left(\frac{\partial \lambda F}{\partial \ddot{x}}\right) = m(\lambda_{tt} + \lambda_{tx}\dot{x} + \lambda_{t\dot{x}}\ddot{x} + \lambda_x\ddot{x} + \lambda_{tx}\dot{x} + \lambda_{xx}(\dot{x})^2 + \lambda_{\dot{x}x}\ddot{x}\dot{x} + \lambda_{\dot{x}t}\ddot{x} + \lambda_{\dot{x}x}\ddot{x}\dot{x} + \lambda_{\dot{x}\dot{x}}(\ddot{x})^2 + \lambda_{\dot{x}}\dddot{x})$$

If the product λF is a total derivative with respect to $'t'$ then, $\xi(\lambda F) = 0$. Assuming λ has the form $\lambda = a(t) + b(t)x + c(t)\dot{x}$ the equality below is obtained.

$$2mb(\dot{x}) - kc(\dot{x}) + kxb - kx(\dot{c}) + ka + kbx + kc(\dot{x}) + m(\ddot{a}) + mx(\dddot{b}) + m(\ddot{c})(\dot{x}) + 2mb(\ddot{x}) + m(\dot{c})(\ddot{x}) = 0.$$

One way to satisfy this equation is to equate the coefficients of $x, \dot{x}, \ddot{x}$ to 0. Consequently, we get the equations:

$$-2b = \dot{c}$$

$$4bk + m\ddot{b} = 0$$

$$ka + m\ddot{a} = 0$$

Solving these ordinary differential equations yield the solutions:

$$a(t) = a_1\cos(w_1 t) + a_2\sin(w_1 t), \qquad w_1 = \sqrt{\frac{k}{m}}$$

$$b(t) = b_1\cos(w_2 t) + b_2\sin(w_2 t), \qquad w_2 = 2\sqrt{\frac{k}{m}}$$

$$c(t) = \frac{2b_2}{w_2}\cos(w_2 t) - \frac{2b_1}{w_2}\sin(w_2 t) + c_3$$

To find one of the conserved quantities we will assume $a_1 = b_1 = b_2 = c_1 = c_2 = 0$ *and*, $a_2 = c_3 = 1$ hence the multiplier will take the form $\lambda = \sin(w_1 t) + \dot{x}$. The product λF will be:

$$m\ddot{x}\sin(w_1 t) + m\ddot{x}\dot{x} + kx\sin(w_1 t) + kx\dot{x}$$

This algebraic expression should be the time derivative of a conserved quantity $\phi(t)$. To verify this we will integrate this expression with respect to time.

$$\int \lambda F\,dt = \int m\ddot{x}\sin(w_1 t)\,dt + \int m\ddot{x}\dot{x}\,dt + \int kx\dot{x}\,dt + \int kx\sin(w_1 t)\,dt \tag{2.1}$$

Integrating by parts the last term of [2.1] twice gives,

$$\int kx\sin(w_1 t)\,dt = -\frac{kx}{w_1}\cos(w_1 t) + \frac{k\dot{x}}{w_1^2}\sin(w_1 t) - \int m\ddot{x}\sin(w_1 t)\,dt.$$

Hence we have,

$$\int \lambda F\,dt = \frac{d}{dt}(\phi(t,x,\ddot{x})) = \frac{d}{dt}\left(\frac{m\dot{x}^2}{2} + \frac{kx^2}{2} - \frac{kx}{w_1}\cos(w_1 t) + \frac{k\dot{x}}{w_1^2}\sin(w_1 t) + d\right).$$

2.3 Example: Kepler's Problem

The equations governing the 2-Body problem in normalized form are,

$$\ddot{q_1} = -\frac{q_1}{(q_1^2+q_2^2)^{3/2}}, \qquad \ddot{q_2} = -\frac{q_2}{(q_1^2+q_2^2)^{3/2}}.$$

These second order equations will be converted to first order equations,

$$\begin{aligned} \dot{u} &= -\tfrac{q_1}{(q_1^2+q_2^2)^{3/2}}, & \dot{q_1} - u = 0,\\ \dot{v} &= -\tfrac{q_2}{(q_1^2+q_2^2)^{3/2}}, & \dot{q_2} - v = 0, \end{aligned}$$

q_1 and q_2 are components of the vector describing the position of second body relative to first body. These equations will be placed in vector $\vec{F}$. Similarly to Harmonic Oscillator case, applying Euler-Lagrange operator to $\vec{\lambda}\cdot\vec{F}$ will result in $\xi(\vec{\lambda}\cdot\vec{F}) = 0$ for $\vec{\lambda}$ which is a conservation law multiplier. In particular we assume the form of $\vec{\lambda}$, 0 if $\vec{\lambda}\cdot\vec{F}$ is a total derivative.

$$\vec{\lambda}(q_1,q_2,u,v,\dot{q_1},\dot{q_2},\dot{u},\dot{v}) = \begin{pmatrix} \lambda_1(q_1,q_2,u,v,\dot{q_1},\dot{q_2},\dot{u},\dot{v})\\ \lambda_2(q_1,q_2,u,v,\dot{q_1},\dot{q_2},\dot{u},\dot{v})\\ \lambda_3(q_1,q_2,u,v,\dot{q_1},\dot{q_2},\dot{u},\dot{v})\\ \lambda_4(q_1,q_2,u,v,\dot{q_1},\dot{q_2},\dot{u},\dot{v}) \end{pmatrix},$$

and write

$$\vec{F}(q_1,q_2,u,v,\dot{q_1},\dot{q_2},\dot{u},\dot{v}) = \begin{pmatrix} \dot{u} + \frac{q_1}{(q_1^2+q_2^2)^{3/2}}\\ \dot{q_1} - u\\ \dot{v} + \frac{q_2}{(q_1^2+q_2^2)^{3/2}}\\ \dot{q_2} - v \end{pmatrix}.$$

This leads to,

$$\vec{\lambda}\cdot\vec{F} = \lambda_1(\dot{u} + \frac{q_1}{(q_1^2+q_2^2)^{3/2}}) + \lambda_2(\dot{q_1} - u) + \lambda_3(\dot{v} + \frac{q_2}{(q_1^2+q_2^2)^{3/2}}) + \lambda_4(\dot{q_2} - v).$$

$\xi(\vec{\lambda}\cdot\vec{F}) = 0$ by letting coefficients next to highest derivatives of q_1, q_2, u, v should be 0. This condition leads to 10 distinct equations, where $\chi_0(q_1,q_2) = \frac{q_1}{(q_1^2+q_2^2)^{3/2}}, \chi_1(q_1,q_2) = \frac{q_2}{(q_1^2+q_2^2)^{3/2}}, \chi_0^0 = \frac{\partial\chi_0}{\partial q_1}, \chi_1^1 = \frac{\partial\chi_1}{\partial q_2}, \chi_0^1 = \chi_1^0 = \frac{\partial\chi_0}{\partial q_2} = \frac{\partial\chi_1}{\partial q_1}$.

$$\left[\frac{\partial^2\lambda_1}{\partial\dot{q_1}\partial\dot{q_2}}(\dot{u}+\chi_0) + \frac{\partial^2\lambda_2}{\partial\dot{q_1}\partial\dot{q_2}}(\dot{q_1}-u) + \frac{\partial^2\lambda_3}{\partial\dot{q_1}\partial\dot{q_2}}(\dot{v}+\chi_1) + \frac{\partial^2\lambda_1}{\partial\dot{q_1}\partial\dot{q_2}}(\dot{q_2}-v) + \frac{\partial\lambda_2}{\partial\dot{q_2}} + \frac{\partial\lambda_4}{\partial\dot{q_1}}\right] = 0,$$

$$\left[\frac{\partial^2\lambda_1}{\partial\dot{q_2}^2}(\dot{u}+\chi_0) + \frac{\partial^2\lambda_2}{\partial\dot{q_2}^2}(\dot{q_1}-u) + \frac{\partial^2\lambda_3}{\partial\dot{q_2}^2}(\dot{v}+\chi_1) + \frac{\partial^2\lambda_1}{\partial\dot{q_2}^2}(\dot{q_2}-v) + \frac{\partial\lambda_4}{\partial\dot{q_2}} + \frac{\partial\lambda_4}{\partial\dot{q_2}}\right] = 0,$$

$$\left[\frac{\partial^2\lambda_1}{\partial\dot{u}\partial\dot{q_2}}(\dot{u}+\chi_0)+\frac{\partial^2\lambda_2}{\partial\dot{u}\partial\dot{q_2}}(\dot{q}_1-u)+\frac{\partial^2\lambda_3}{\partial\dot{u}\partial\dot{q_2}}(\dot{v}+\chi_1)+\frac{\partial^2\lambda_1}{\partial\dot{u}\partial\dot{q_2}}(\dot{q}_2-v)+\frac{\partial\lambda_1}{\partial\dot{q_2}}+\frac{\partial\lambda_4}{\partial\dot{u}}\right]=0,$$

$$\left[\frac{\partial^2\lambda_1}{\partial\dot{v}\partial\dot{q_2}}(\dot{u}+\chi_0)+\frac{\partial^2\lambda_2}{\partial\dot{v}\partial\dot{q_2}}(\dot{q}_1-u)+\frac{\partial^2\lambda_3}{\partial\dot{v}\partial\dot{q_2}}(\dot{v}+\chi_1)+\frac{\partial^2\lambda_1}{\partial\dot{v}\partial\dot{q_2}}(\dot{q}_2-v)+\frac{\partial\lambda_3}{\partial\dot{q_2}}+\frac{\partial\lambda_4}{\partial\dot{v}}\right]=0,$$

$$\left[\frac{\partial^2\lambda_1}{\partial\dot{q_1}^2}(\dot{u}+\chi_0)+\frac{\partial^2\lambda_2}{\partial\dot{q_1}^2}(\dot{q}_1-u)+\frac{\partial^2\lambda_3}{\partial\dot{q_1}^2}(\dot{v}+\chi_1)+\frac{\partial^2\lambda_1}{\partial\dot{q_1}^2}(\dot{q}_2-v)+\frac{\partial\lambda_2}{\partial\dot{q_1}}+\frac{\partial\lambda_2}{\partial\dot{q_1}}\right]=0,$$

$$\left[\frac{\partial^2\lambda_1}{\partial\dot{q_1}\partial\dot{u}}(\dot{u}+\chi_0)+\frac{\partial^2\lambda_2}{\partial\dot{q_1}\partial\dot{u}}(\dot{q}_1-u)+\frac{\partial^2\lambda_3}{\partial\dot{q_1}\partial\dot{u}}(\dot{v}+\chi_1)+\frac{\partial^2\lambda_1}{\partial\dot{q_1}\partial\dot{u}}(\dot{q}_2-v)+\frac{\partial\lambda_2}{\partial\dot{u}}+\frac{\partial\lambda_1}{\partial\dot{q_1}}\right]=0, \quad (2.2)$$

$$\left[\frac{\partial^2\lambda_1}{\partial\dot{q_1}\partial\dot{v}}(\dot{u}+\chi_0)+\frac{\partial^2\lambda_2}{\partial\dot{q_1}\partial\dot{v}}(\dot{q}_1-u)+\frac{\partial^2\lambda_3}{\partial\dot{q_1}\partial\dot{v}}(\dot{v}+\chi_1)+\frac{\partial^2\lambda_1}{\partial\dot{q_1}\partial\dot{v}}(\dot{q}_2-v)+\frac{\partial\lambda_2}{\partial\dot{v}}+\frac{\partial\lambda_3}{\partial\dot{q_1}}\right]=0,$$

$$\left[\frac{\partial^2\lambda_1}{\partial\dot{u}^2}(\dot{u}+\chi_0)+\frac{\partial^2\lambda_2}{\partial\dot{u}^2}(\dot{q}_1-u)+\frac{\partial^2\lambda_3}{\partial\dot{u}^2}(\dot{v}+\chi_1)+\frac{\partial^2\lambda_1}{\partial\dot{u}^2}(\dot{q}_2-v)+\frac{\partial\lambda_1}{\partial\dot{u}}+\frac{\partial\lambda_1}{\partial\dot{u}}\right]=0,$$

$$\left[\frac{\partial^2\lambda_1}{\partial\dot{v}^2}(\dot{u}+\chi_0)+\frac{\partial^2\lambda_2}{\partial\dot{v}^2}(\dot{q}_1-u)+\frac{\partial^2\lambda_3}{\partial\dot{v}^2}(\dot{v}+\chi_1)+\frac{\partial^2\lambda_1}{\partial\dot{v}^2}(\dot{q}_2-v)+\frac{\partial\lambda_3}{\partial\dot{v}}+\frac{\partial\lambda_3}{\partial\dot{v}}\right]=0,$$

$$\left[\frac{\partial^2\lambda_1}{\partial\dot{u}\partial\dot{v}}(\dot{u}+\chi_0)+\frac{\partial^2\lambda_2}{\partial\dot{u}\partial\dot{v}}(\dot{q}_1-u)+\frac{\partial^2\lambda_3}{\partial\dot{u}\partial\dot{v}}(\dot{v}+\chi_1)+\frac{\partial^2\lambda_1}{\partial\dot{u}\partial\dot{v}}(\dot{q}_2-v)+\frac{\partial\lambda_1}{\partial\dot{v}}+\frac{\partial\lambda_3}{\partial\dot{u}}\right]=0,$$

In general, equations [2.2] is difficult to solve. Hence we assume to find a particular soluton of [2.2] by assuming a linear form in $\vec{\lambda}$,

$$\vec{\lambda}=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ u \\ v \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{u} \\ \dot{v} \end{pmatrix}.$$

Assuming $\vec{\lambda}$ is a linear vector function, the Euler-Lagrange operator will yield four equations,

$$\begin{gathered}\dot{u}a_{11}+\lambda_1\chi_0^0+a_{11}\chi_0+a_{21}\dot{q}_1-a_{21}u+a_{31}\dot{v}+a_{31}\chi_1+\lambda_3\chi_1^0+a_{41}\dot{q}_2-a_{41}v- \\ a_{15}(\ddot{u}+\chi_0^0\dot{q}_1+\chi_0^1\dot{q}_2)-a_{25}(\ddot{q}_1-\dot{u})-(a_{21}\dot{q}_1+a_{22}\dot{q}_2+a_{23}\dot{u}+a_{24}\dot{v}+a_{25}\ddot{q}_1+ \\ a_{26}\ddot{q}_2+a_{27}\ddot{u}+a_{28}\ddot{v})-a_{35}(\ddot{v}+\chi_1^0\dot{q}_1+\chi_1^1\dot{q}_2)-a_{45}(\ddot{q}_2-\dot{v})=0\end{gathered}$$

$$\begin{gathered}\dot{u}a_{12}+\lambda_1\chi_0^1+a_{12}\chi_0+a_{22}\dot{q}_1-a_{22}u+a_{32}\dot{v}+a_{32}\chi_1+\lambda_3\chi_1^1+a_{42}\dot{q}_2-a_{42}v- \\ a_{16}(\ddot{u}+\chi_0^0\dot{q}_1+\chi_0^1\dot{q}_2)-a_{26}(\ddot{q}_1-\dot{u})-(a_{41}\dot{q}_1+a_{42}\dot{q}_2+a_{43}\dot{u}+a_{44}\dot{v}+a_{45}\ddot{q}_1+ \\ a_{46}\ddot{q}_2+a_{47}\ddot{u}+a_{48}\ddot{v})-a_{36}(\ddot{v}+\chi_1^0\dot{q}_1+\chi_1^1\dot{q}_2)-a_{46}(\ddot{q}_2-\dot{v})=0\end{gathered}$$

$$\dot{u}a_{13} - \lambda_2 + a_{13}\chi_0 + a_{23}\dot{q}_1 - a_{23}u + a_{33}\dot{v} + a_{33}\chi_1 + a_{43}\dot{q}_2 - a_{43}v -$$
$$a_{17}(\ddot{u} + \chi_0^0\dot{q}_1 + \chi_0^1\dot{q}_2) - a_{27}(\ddot{q}_1 - \dot{u}) - (a_{11}\dot{q}_1 + a_{12}\dot{q}_2 + a_{13}\dot{u} + a_{14}\dot{v} + a_{15}\ddot{q}_1 +$$
$$a_{16}\ddot{q}_2 + a_{17}\ddot{u} + a_{18}\ddot{v}) - a_{37}(\ddot{v} + \chi_1^0\dot{q}_1 + \chi_1^1\dot{q}_2) - a_{47}(\ddot{q}_2 - \dot{v}) = 0$$

$$\dot{u}a_{14} - \lambda_4 + a_{14}\chi_0 + a_{24}\dot{q}_1 - a_{24}u + a_{34}\dot{v} + a_{34}\chi_1 + a_{44}\dot{q}_2 - a_{44}v -$$
$$a_{18}(\ddot{u} + \chi_0^0\dot{q}_1 + \chi_0^1\dot{q}_2) - a_{28}(\ddot{q}_1 - \dot{u}) - (a_{31}\dot{q}_1 + a_{32}\dot{q}_2 + a_{33}\dot{u} + a_{34}\dot{v} + a_{35}\ddot{q}_1 +$$
$$a_{36}\ddot{q}_2 + a_{37}\ddot{u} + a_{38}\ddot{v}) - a_{38}(\ddot{v} + \chi_1^0\dot{q}_1 + \chi_1^1\dot{q}_2) - a_{48}(\ddot{q}_2 - \dot{v}) = 0$$

If the coefficients of $q_1, q_2, u, v, \dot{q}_1, \dot{q}_2, \dot{u}$ and $\dot{v}$ in the equations are 0 then equations are satisfied.This results in all a_{ij} being 0 except a_{15}, a_{27}, a_{36} and a_{48}, where $a_{15} = -a_{27} = a_{36} = -a_{48}$. Hence the multiplier takes the form, where K is a constant,

$$\vec{\lambda} = K \begin{pmatrix} \dot{q}_1 \\ -\dot{u} \\ \dot{q}_2 \\ \dot{v} \end{pmatrix}.$$

It now can be verified that:

$$\vec{\lambda} \cdot \vec{F} = \dot{q}_1(\dot{u} + \frac{q_1}{(q_1^2 + q_2^2)^{3/2}}) - \dot{u}(\dot{q}_1 - u) + \dot{q}_2(\dot{v} + \frac{q_2}{(q_1^2 + q_2^2)^{3/2}}) - \dot{v}(\dot{q}_2 - v) = D_t(\Phi),$$

where

$$\phi = \frac{u^2}{2} + \frac{v^2}{2} - \frac{1}{\sqrt{q_1^2 + q_2^2}}.$$

2.4 Example: Point Vortex Problem

Interaction of two point vortices on a plane is ruled by system of four non-linear equations such that,

$$\vec{F} = \begin{pmatrix} \dot{x}_1 + \frac{1}{2\pi}\frac{\Gamma_2(y_1 - y_2)}{r_{1,2}^2} \\ \dot{y}_1 - \frac{1}{2\pi}\frac{\Gamma_2(x_1 - x_2)}{r_{1,2}^2} \\ \dot{x}_2 + \frac{1}{2\pi}\frac{\Gamma_1(y_2 - y_1)}{r_{1,2}^2} \\ \dot{y}_2 - \frac{1}{2\pi}\frac{\Gamma_1(x_2 - x_1)}{r_{1,2}^2} \end{pmatrix} = 0.$$

In the equation, (x_1, y_1) and (x_2, y_2) are instantaneous positions of first and second vortices respectively. Moreover, $r_{1,2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ is the distance between two vortices . Assuming $\lambda = \lambda(x_1, y_1, x_2, y_2, \dot{x}_1, \dot{y}_1, \dot{x}_2, \dot{y}_2)$ and executing $\vec{\lambda} \cdot \vec{F}$ leads to,

$$\lambda \cdot F = \lambda_1\dot{x}_1 + \lambda_1\frac{1}{2\pi}\frac{\Gamma_2\,(y_1 - y_2)}{r_{1,2}^2} + \lambda_2\dot{y}_1 - \lambda_2\frac{1}{2\pi}\frac{\Gamma_2\,(x_1 - x_2)}{r_{1,2}^2} + \lambda_3\dot{x}_2 + \lambda_3\frac{1}{2\pi}\frac{\Gamma_1\,(y_2 - y_1)}{r_{1,2}^2} + \lambda_4\dot{y}_2 - \lambda_4\frac{1}{2\pi}\frac{\Gamma_1\,(x_2 - x_1)}{r_{1,2}^2}. \quad (2.3)$$

For simplicity certain substitutions will be done. Such as ,

$$\rho_1 = \frac{1}{2\pi}\frac{(y_1 - y_2)}{r_{1,2}^2}, \quad \rho_2 = \frac{1}{2\pi}\frac{(x_1 - x_2)}{r_{1,2}^2}, \quad \rho_1^1 = \frac{\partial\rho_1}{\partial x_1}, \quad \rho_1^2 = \frac{\partial\rho_1}{\partial y_1}, \quad \rho_1^3 = \frac{\partial\rho_1}{\partial x_2}, \quad \rho_1^4 = \frac{\partial\rho_1}{\partial y_2},$$
$$\rho_2^1 = \frac{\partial\rho_2}{\partial x_1}, \quad \rho_2^2 = \frac{\partial\rho_2}{\partial y_1}, \quad \rho_2^3 = \frac{\partial\rho_2}{\partial x_2}, \quad \rho_2^4 = \frac{\partial\rho_2}{\partial y_2}.$$

Nature of the functions ρ_1 and ρ_2 induces,

$$\rho_1^1 = -\rho_1^3 = -\rho_2^2 = \rho_2^4, \qquad \rho_1^2 = -\rho_1^4 = \rho_2^1 = -\rho_2^3.$$

Similar to previous examples applying Euler-Lagrange Operator to $\vec{\lambda} \cdot \vec{F}$ yields four equations such that,

$$\vec{\xi}\left(\vec{\lambda}\cdot\vec{F}\right) = \begin{pmatrix} \frac{\partial(\vec{\lambda}\cdot\vec{F})}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial(\vec{\lambda}\cdot\vec{F})}{\partial \dot{x}_1}\right) \\ \frac{\partial(\vec{\lambda}\cdot\vec{F})}{\partial y_1} - \frac{d}{dt}\left(\frac{\partial(\vec{\lambda}\cdot\vec{F})}{\partial \dot{y}_1}\right) \\ \frac{\partial(\vec{\lambda}\cdot\vec{F})}{\partial x_2} - \frac{d}{dt}\left(\frac{\partial(\vec{\lambda}\cdot\vec{F})}{\partial \dot{x}_2}\right) \\ \frac{\partial(\vec{\lambda}\cdot\vec{F})}{\partial y_2} - \frac{d}{dt}\left(\frac{\partial(\vec{\lambda}\cdot\vec{F})}{\partial \dot{y}_2}\right) \end{pmatrix} = 0,$$

where,

$$\frac{\partial\left(\vec{\lambda}\cdot\vec{F}\right)}{\partial x_1} = \dot{x}_1\frac{\partial\lambda_1}{\partial x_1} + \Gamma_2\rho_2^1\lambda_1 + \Gamma_2\rho_2\frac{\partial\lambda_1}{\partial x_1} + \dot{y}_1\frac{\partial\lambda_2}{\partial x_1} - \Gamma_2\rho_1^1\lambda_2 - \Gamma_2\rho_1\frac{\partial\lambda_2}{\partial x_1} + \dot{x}_2\frac{\partial\lambda_3}{\partial x_1} - \Gamma_1\rho_2^1\lambda_3 - \Gamma_1\rho_2\frac{\partial\lambda_3}{\partial x_1} + \dot{y}_2\frac{\partial\lambda_4}{\partial x_1} + \Gamma_1\rho_1^1\lambda_4 + \Gamma_2\rho_1\frac{\partial\lambda_4}{\partial x_1},$$

$$\frac{\partial\left(\vec{\lambda}\cdot\vec{F}\right)}{\partial y_1} = \dot{x}_1\frac{\partial\lambda_1}{\partial y_1} + \Gamma_2\rho_2^2\lambda_1 + \Gamma_2\rho_2\frac{\partial\lambda_1}{\partial y_1} + \dot{y}_1\frac{\partial\lambda_2}{\partial y_1} - \Gamma_2\rho_1^2\lambda_2 - \Gamma_2\rho_1\frac{\partial\lambda_2}{\partial y_1} + \dot{x}_2\frac{\partial\lambda_3}{\partial y_1} - \Gamma_1\rho_2^2\lambda_3 - \Gamma_1\rho_2\frac{\partial\lambda_3}{\partial y_1} + \dot{y}_2\frac{\partial\lambda_4}{\partial y_1} + \Gamma_1\rho_1^2\lambda_4 + \Gamma_2\rho_1\frac{\partial\lambda_4}{\partial y_1},$$

$$\frac{\partial\left(\vec{\lambda}\cdot\vec{F}\right)}{\partial x_2} = \dot{x}_1\frac{\partial\lambda_1}{\partial x_2} + \Gamma_2\rho_2^3\lambda_1 + \Gamma_2\rho_2\frac{\partial\lambda_1}{\partial x_2} + \dot{y}_1\frac{\partial\lambda_2}{\partial x_2} - \Gamma_2\rho_1^3\lambda_2 - \Gamma_2\rho_1\frac{\partial\lambda_2}{\partial x_2} + \dot{x}_2\frac{\partial\lambda_3}{\partial x_2} - \Gamma_1\rho_2^3\lambda_3 - \Gamma_1\rho_2\frac{\partial\lambda_3}{\partial x_2} + \dot{y}_2\frac{\partial\lambda_4}{\partial x_2} + \Gamma_1\rho_1^3\lambda_4 + \Gamma_2\rho_1\frac{\partial\lambda_4}{\partial x_2},$$

$$\frac{\partial\left(\vec{\lambda}\cdot\vec{F}\right)}{\partial y_2} = \dot{x}_1\frac{\partial\lambda_1}{\partial y_2} + \Gamma_2\rho_2^4\lambda_1 + \Gamma_2\rho_2\frac{\partial\lambda_1}{\partial y_2} + \dot{y}_1\frac{\partial\lambda_2}{\partial y_2} - \Gamma_2\rho_1^4\lambda_2 - \Gamma_2\rho_1\frac{\partial\lambda_2}{\partial y_2} + \dot{x}_2\frac{\partial\lambda_3}{\partial y_2} - \Gamma_1\rho_2^4\lambda_3 - \Gamma_1\rho_2\frac{\partial\lambda_3}{\partial y_2} + \dot{y}_2\frac{\partial\lambda_4}{\partial y_2} + \Gamma_1\rho_1^4\lambda_4 + \Gamma_2\rho_1\frac{\partial\lambda_4}{\partial y_2},$$

$$\frac{\partial\left(\vec{\lambda}\cdot\vec{F}\right)}{\partial \dot{x}_1} = \frac{\partial\lambda_1}{\partial \dot{x}_1}\left(\dot{x}_1 + \Gamma_2\rho_2\right) + \frac{\partial\lambda_2}{\partial \dot{x}_1}\left(\dot{y}_1 - \Gamma_2\rho_1\right)\frac{\partial\lambda_3}{\partial \dot{x}_1}\left(\dot{x}_2 - \Gamma_1\rho_2\right) + \frac{\partial\lambda_4}{\partial \dot{x}_1}\left(\dot{y}_2 + \Gamma_1\rho_1\right) + \lambda_1,$$

$$\frac{\partial\left(\vec{\lambda}\cdot\vec{F}\right)}{\partial \dot{x}_2} = \frac{\partial\lambda_1}{\partial \dot{x}_2}\left(\dot{x}_1 + \Gamma_2\rho_2\right) + \frac{\partial\lambda_2}{\partial \dot{x}_2}\left(\dot{y}_1 - \Gamma_2\rho_1\right)\frac{\partial\lambda_3}{\partial \dot{x}_2}\left(\dot{x}_2 - \Gamma_1\rho_2\right) + \frac{\partial\lambda_4}{\partial \dot{x}_2}\left(\dot{y}_2 + \Gamma_1\rho_1\right) + \lambda_3,$$

$$\frac{\partial\left(\vec{\lambda}\cdot\vec{F}\right)}{\partial\dot{y}_1}=\frac{\partial\lambda_1}{\partial\dot{y}_1}\left(\dot{x}_1+\Gamma_2\rho_2\right)+\frac{\partial\lambda_2}{\partial\dot{y}_1}\left(\dot{y}_1-\Gamma_2\rho_1\right)\frac{\partial\lambda_3}{\partial\dot{y}_1}\left(\dot{x}_2-\Gamma_1\rho_2\right)+\frac{\partial\lambda_4}{\partial\dot{y}_1}\left(\dot{y}_2+\Gamma_1\rho_1\right)+\lambda_2,$$

$$\frac{\partial\left(\vec{\lambda}\cdot\vec{F}\right)}{\partial\dot{y}_2}=\frac{\partial\lambda_1}{\partial\dot{y}_2}\left(\dot{x}_1+\Gamma_2\rho_2\right)+\frac{\partial\lambda_2}{\partial\dot{y}_2}\left(\dot{y}_1-\Gamma_2\rho_1\right)\frac{\partial\lambda_3}{\partial\dot{y}_2}\left(\dot{x}_2-\Gamma_1\rho_2\right)+\frac{\partial\lambda_4}{\partial\dot{y}_2}\left(\dot{y}_2+\Gamma_1\rho_1\right)+\lambda_4.$$

Like the case of gravitational 2-body problem, in order to satisfy the four equations one can equate the terms next to highest order derivatives($\ddot{x}_1,\ddot{y}_1,\ddot{x}_2,\ddot{y}_2$) to zero. This yields fourteen equations in which ten of them are distinct.Such that,

$$\frac{\partial^2\lambda_1}{\partial\dot{x}_1^2}\left(\dot{x}_1+\Gamma_2\rho_2\right)+\frac{\partial^2\lambda_2}{\partial\dot{x}_1^2}\left(\dot{y}_1-\Gamma_2\rho_1\right)\frac{\partial^2\lambda_3}{\partial\dot{x}_1^2}\left(\dot{x}_2-\Gamma_1\rho_2\right)+\frac{\partial^2\lambda_4}{\partial\dot{x}_1^2}\left(\dot{y}_2+\Gamma_1\rho_1\right)+\frac{\partial\lambda_1}{\partial\dot{x}_1}+\frac{\partial\lambda_1}{\partial\dot{x}_1}=0,$$

$$\frac{\partial^2\lambda_1}{\partial\dot{x}_1\partial\dot{x}_2}\left(\dot{x}_1+\Gamma_2\rho_2\right)+\frac{\partial^2\lambda_2}{\partial\dot{x}_1\partial\dot{x}_2}\left(\dot{y}_1-\Gamma_2\rho_1\right)\frac{\partial^2\lambda_3}{\partial\dot{x}_1\partial\dot{x}_2}\left(\dot{x}_2-\Gamma_1\rho_2\right)+\frac{\partial^2\lambda_4}{\partial\dot{x}_1\partial\dot{x}_2}\left(\dot{y}_2+\Gamma_1\rho_1\right)+\frac{\partial\lambda_1}{\partial\dot{x}_2}+\frac{\partial\lambda_3}{\partial\dot{x}_1}=0,$$

$$\frac{\partial^2\lambda_1}{\partial\dot{x}_2^2}\left(\dot{x}_1+\Gamma_2\rho_2\right)+\frac{\partial^2\lambda_2}{\partial\dot{x}_2^2}\left(\dot{y}_1-\Gamma_2\rho_1\right)\frac{\partial^2\lambda_3}{\partial\dot{x}_2^2}\left(\dot{x}_2-\Gamma_1\rho_2\right)+\frac{\partial^2\lambda_4}{\partial\dot{x}_2^2}\left(\dot{y}_2+\Gamma_1\rho_1\right)+\frac{\partial\lambda_3}{\partial\dot{x}_2}+\frac{\partial\lambda_3}{\partial\dot{x}_2}=0,$$

$$\frac{\partial^2\lambda_1}{\partial\dot{y}_1^2}\left(\dot{x}_1+\Gamma_2\rho_2\right)+\frac{\partial^2\lambda_2}{\partial\dot{y}_1^2}\left(\dot{y}_1-\Gamma_2\rho_1\right)\frac{\partial^2\lambda_3}{\partial\dot{y}_1^2}\left(\dot{x}_2-\Gamma_1\rho_2\right)+\frac{\partial^2\lambda_4}{\partial\dot{y}_1^2}\left(\dot{y}_2+\Gamma_1\rho_1\right)+\frac{\partial\lambda_2}{\partial\dot{y}_1}+\frac{\partial\lambda_2}{\partial\dot{y}_1}=0,$$

$$\frac{\partial^2\lambda_1}{\partial\dot{y}_2^2}\left(\dot{x}_1+\Gamma_2\rho_2\right)+\frac{\partial^2\lambda_2}{\partial\dot{y}_2^2}\left(\dot{y}_1-\Gamma_2\rho_1\right)\frac{\partial^2\lambda_3}{\partial\dot{y}_2^2}\left(\dot{x}_2-\Gamma_1\rho_2\right)+\frac{\partial^2\lambda_4}{\partial\dot{y}_2^2}\left(\dot{y}_2+\Gamma_1\rho_1\right)+\frac{\partial\lambda_4}{\partial\dot{y}_2}+\frac{\partial\lambda_4}{\partial\dot{y}_2}=0,$$

$$\frac{\partial^2\lambda_1}{\partial\dot{y}_1\partial\dot{y}_2}\left(\dot{x}_1+\Gamma_2\rho_2\right)+\frac{\partial^2\lambda_2}{\partial\dot{y}_1\partial\dot{y}_2}\left(\dot{y}_1-\Gamma_2\rho_1\right)\frac{\partial^2\lambda_3}{\partial\dot{y}_1\partial\dot{y}_2}\left(\dot{x}_2-\Gamma_1\rho_2\right)+\frac{\partial^2\lambda_4}{\partial\dot{y}_1\partial\dot{y}_2}\left(\dot{y}_2+\Gamma_1\rho_1\right)+\frac{\partial\lambda_2}{\partial\dot{y}_2}+\frac{\partial\lambda_4}{\partial\dot{y}_1}=0,$$

$$\frac{\partial^2\lambda_1}{\partial\dot{y}_1\partial\dot{x}_1}\left(\dot{x}_1+\Gamma_2\rho_2\right)+\frac{\partial^2\lambda_2}{\partial\dot{y}_1\partial\dot{x}_1}\left(\dot{y}_1-\Gamma_2\rho_1\right)\frac{\partial^2\lambda_3}{\partial\dot{x}_1\partial\dot{y}_1}\left(\dot{x}_1-\Gamma_1\rho_2\right)+\frac{\partial^2\lambda_4}{\partial\dot{x}_1\partial\dot{y}_1}\left(\dot{y}_2+\Gamma_1\rho_1\right)+\frac{\partial\lambda_2}{\partial\dot{x}_1}+\frac{\partial\lambda_1}{\partial\dot{y}_1}=0,$$

$$\frac{\partial^2\lambda_1}{\partial\dot{y}_1\partial\dot{x}_2}\left(\dot{x}_1+\Gamma_2\rho_2\right)+\frac{\partial^2\lambda_2}{\partial\dot{y}_1\partial\dot{x}_2}\left(\dot{y}_1-\Gamma_2\rho_1\right)\frac{\partial^2\lambda_3}{\partial\dot{x}_2\partial\dot{y}_1}\left(\dot{x}_1-\Gamma_1\rho_2\right)+\frac{\partial^2\lambda_4}{\partial\dot{x}_2\partial\dot{y}_1}\left(\dot{y}_2+\Gamma_1\rho_1\right)+\frac{\partial\lambda_2}{\partial\dot{x}_2}+\frac{\partial\lambda_3}{\partial\dot{y}_1}=0,$$

$$\frac{\partial^2\lambda_1}{\partial\dot{y}_2\partial\dot{x}_2}\left(\dot{x}_1+\Gamma_2\rho_2\right)+\frac{\partial^2\lambda_2}{\partial\dot{y}_2\partial\dot{x}_2}\left(\dot{y}_2-\Gamma_2\rho_1\right)\frac{\partial^2\lambda_3}{\partial\dot{x}_2\partial\dot{y}_2}\left(\dot{x}_1-\Gamma_1\rho_2\right)+\frac{\partial^2\lambda_4}{\partial\dot{x}_2\partial\dot{y}_2}\left(\dot{y}_2+\Gamma_1\rho_1\right)+\frac{\partial\lambda_3}{\partial\dot{y}_2}+\frac{\partial\lambda_4}{\partial\dot{x}_2}=0,$$

$$\frac{\partial^2\lambda_1}{\partial\dot{y}_2\partial\dot{x}_1}\left(\dot{x}_1+\Gamma_2\rho_2\right)+\frac{\partial^2\lambda_2}{\partial\dot{y}_2\partial\dot{x}_1}\left(\dot{y}_2-\Gamma_2\rho_1\right)\frac{\partial^2\lambda_3}{\partial\dot{x}_1\partial\dot{y}_2}\left(\dot{x}_1-\Gamma_1\rho_2\right)+\frac{\partial^2\lambda_4}{\partial\dot{x}_1\partial\dot{y}_2}\left(\dot{y}_2+\Gamma_1\rho_1\right)+\frac{\partial\lambda_1}{\partial\dot{y}_2}+\frac{\partial\lambda_4}{\partial\dot{x}_1}=0.$$

These equations form a system of partial differential equations.One of the solutions for this system is a linear function of λ and its variables. Thus for simplicity we will seek a linear multiplier λ such that,

$$\vec{\lambda} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_2 \\ \dot{y}_2 \\ 1 \end{pmatrix}.$$

Consequently, four equations resulting from Euler-Lagrange operator are,

$$\begin{aligned} &a_{11}\dot{x}_1+\Gamma_2\rho_2a_{11}+\Gamma_2\rho_2^1\lambda_1+a_{21}\dot{y}_1-\Gamma_2\rho_1a_{21}-\Gamma_2\rho_1^1\lambda_2+a_{31}\dot{x}_2-\Gamma_1\rho_2a_{31}-\Gamma_1\rho_2^1\lambda_3+a_{41}\dot{y}_2+\Gamma_1\rho_1a_{41}+\Gamma_1\rho_1^1\lambda_4 \\ &-(a_{11}\dot{x}_1+a_{12}\dot{y}_1+a_{13}\dot{x}_2+a_{14}\dot{y}_2+a_{15}\ddot{x}_1+a_{16}\ddot{y}_1+a_{17}\ddot{x}_2+a_{18}\ddot{y}_2)-a_{15}(\ddot{x}_1+\Gamma_2(\rho_2^1\dot{x}_1+\rho_2^2\dot{y}_1+\rho_2^3\dot{x}_2+\rho_2^4\dot{y}_2)) \\ &-a_{25}(\ddot{y}_1-\Gamma_2(\rho_1^1\dot{x}_1+\rho_1^2\dot{y}_1+\rho_1^3\dot{x}_2+\rho_1^4\dot{y}_2))-a_{35}(\ddot{x}_2-\Gamma_1(\rho_2^1\dot{x}_1+\rho_2^2\dot{y}_1+\rho_2^3\dot{x}_2+\rho_2^4\dot{y}_2))-a_{45}(\ddot{y}_2+ \\ &\Gamma_1(\rho_1^1\dot{x}_1+\rho_1^2\dot{y}_1+\rho_1^3\dot{x}_2+\rho_1^4\dot{y}_2))=0, \end{aligned}$$

$$\begin{aligned} &a_{12}\dot{x}_1+\Gamma_2\rho_2a_{12}+\Gamma_2\rho_2^2\lambda_1+a_{22}\dot{y}_1-\Gamma_2\rho_1a_{22}-\Gamma_2\rho_1^2\lambda_2+a_{32}\dot{x}_2-\Gamma_1\rho_2a_{32}-\Gamma_1\rho_2^2\lambda_3+a_{42}\dot{y}_2+\Gamma_1\rho_1a_{42}+\Gamma_1\rho_1^2\lambda_4 \\ &-(a_{21}\dot{x}_1+a_{22}\dot{y}_1+a_{23}\dot{x}_2+a_{24}\dot{y}_2+a_{25}\ddot{x}_1+a_{26}\ddot{y}_1+a_{27}\ddot{x}_2+a_{28}\ddot{y}_2)-a_{16}(\ddot{x}_1+\Gamma_2(\rho_2^1\dot{x}_1+\rho_2^2\dot{y}_1+\rho_2^3\dot{x}_2+\rho_2^4\dot{y}_2)) \\ &-a_{26}(\ddot{y}_1-\Gamma_2(\rho_1^1\dot{x}_1+\rho_1^2\dot{y}_1+\rho_1^3\dot{x}_2+\rho_1^4\dot{y}_2))-a_{36}(\ddot{x}_2-\Gamma_1(\rho_2^1\dot{x}_1+\rho_2^2\dot{y}_1+\rho_2^3\dot{x}_2+\rho_2^4\dot{y}_2))-a_{46}(\ddot{y}_2+ \\ &\Gamma_1(\rho_1^1\dot{x}_1+\rho_1^2\dot{y}_1+\rho_1^3\dot{x}_2+\rho_1^4\dot{y}_2))=0, \end{aligned}$$

$$\begin{aligned} &a_{13}\dot{x}_1+\Gamma_2\rho_2a_{13}+\Gamma_2\rho_2^3\lambda_1+a_{23}\dot{y}_1-\Gamma_2\rho_1a_{23}-\Gamma_2\rho_1^3\lambda_2+a_{33}\dot{x}_2-\Gamma_1\rho_2a_{33}-\Gamma_1\rho_2^3\lambda_3+a_{43}\dot{y}_2+\Gamma_1\rho_1a_{43}+\Gamma_1\rho_1^3\lambda_4 \\ &-(a_{31}\dot{x}_1+a_{32}\dot{y}_1+a_{33}\dot{x}_2+a_{34}\dot{y}_2+a_{35}\ddot{x}_1+a_{36}\ddot{y}_1+a_{37}\ddot{x}_2+a_{38}\ddot{y}_2)-a_{17}(\ddot{x}_1+\Gamma_2(\rho_2^1\dot{x}_1+\rho_2^2\dot{y}_1+\rho_2^3\dot{x}_2+\rho_2^4\dot{y}_2)) \\ &-a_{27}(\ddot{y}_1-\Gamma_2(\rho_1^1\dot{x}_1+\rho_1^2\dot{y}_1+\rho_1^3\dot{x}_2+\rho_1^4\dot{y}_2))-a_{37}(\ddot{x}_2-\Gamma_1(\rho_2^1\dot{x}_1+\rho_2^2\dot{y}_1+\rho_2^3\dot{x}_2+\rho_2^4\dot{y}_2))-a_{47}(\ddot{y}_2 \\ &+\Gamma_1(\rho_1^1\dot{x}_1+\rho_1^2\dot{y}_1+\rho_1^3\dot{x}_2+\rho_1^4\dot{y}_2))=0, \end{aligned}$$

$$\begin{aligned} &a_{14}\dot{x}_1+\Gamma_2\rho_2a_{14}+\Gamma_2\rho_2^4\lambda_1+a_{24}\dot{y}_1-\Gamma_2\rho_1a_{24}-\Gamma_2\rho_1^4\lambda_2+a_{34}\dot{x}_2-\Gamma_1\rho_2a_{34}-\Gamma_1\rho_2^4\lambda_3+a_{44}\dot{y}_2+\Gamma_1\rho_1a_{44}+\Gamma_1\rho_1^4\lambda_4 \\ &-(a_{41}\dot{x}_1+a_{42}\dot{y}_1+a_{43}\dot{x}_2+a_{44}\dot{y}_2+a_{45}\ddot{x}_1+a_{46}\ddot{y}_1+a_{47}\ddot{x}_2+a_{48}\ddot{y}_2)-a_{18}(\ddot{x}_1+\Gamma_2(\rho_2^1\dot{x}_1+\rho_2^2\dot{y}_1+\rho_2^3\dot{x}_2+\rho_2^4\dot{y}_2)) \\ &-a_{28}(\ddot{y}_1-\Gamma_2(\rho_1^1\dot{x}_1+\rho_1^2\dot{y}_1+\rho_1^3\dot{x}_2+\rho_1^4\dot{y}_2))-a_{38}(\ddot{x}_2-\Gamma_1(\rho_2^1\dot{x}_1+\rho_2^2\dot{y}_1+\rho_2^3\dot{x}_2+\rho_2^4\dot{y}_2))-a_{48}(\ddot{y}_2 \\ &\Gamma_1(\rho_1^1\dot{x}_1+\rho_1^2\dot{y}_1+\rho_1^3\dot{x}_2+\rho_1^4\dot{y}_2))=0, \end{aligned}$$

By solving the four equations above for variables $a_{ij}\, i = 1, 2\ldots\; j = 1, 2\ldots$ one can obtain a non-unique conservation law multiplier.Although there is more than one solution, one of the solutions is given by,

$$\vec{\lambda} = \begin{pmatrix} 2\Gamma_1 x_1 + \Gamma_1 \dot{y}_1 + \Gamma_1 \\ 2\Gamma_1 x_1 - \Gamma_1 \dot{x}_1 + \Gamma_1 \\ 2\Gamma_2 x_1 + \Gamma_2 \dot{y}_2 + \Gamma_2 \\ 2\Gamma_2 x_1 - \Gamma_2 \dot{x}_2 + \Gamma_2 \end{pmatrix}.$$

λ can be decomposed in to four different vectors each of which act as a conservation law multiplier as well such that,

$$\vec{\lambda} = \begin{pmatrix} 2\Gamma_1 x_1 + \Gamma_1 \dot{y}_1 + \Gamma_1 \\ 2\Gamma_1 y_1 - \Gamma_1 \dot{x}_1 + \Gamma_1 \\ 2\Gamma_2 x_2 + \Gamma_2 \dot{y}_2 + \Gamma_2 \\ 2\Gamma_2 y_2 - \Gamma_2 \dot{x}_2 + \Gamma_2 \end{pmatrix} = \begin{pmatrix} 2\Gamma_1 x_1 \\ 2\Gamma_1 y_1 \\ 2\Gamma_2 x_2 \\ 2\Gamma_2 y_2 \end{pmatrix} + \begin{pmatrix} \Gamma_1 \dot{y}_1 \\ -\Gamma_1 \dot{x}_1 \\ \Gamma_2 \dot{y}_2 \\ -\Gamma_2 \dot{x}_2 \end{pmatrix} + \begin{pmatrix} \Gamma_1 \\ 0 \\ \Gamma_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \Gamma_1 \\ 0 \\ \Gamma_2 \end{pmatrix},$$

$$\vec{\lambda_L} = \begin{pmatrix} 2\Gamma_1 x_1 \\ 2\Gamma_1 y_1 \\ 2\Gamma_2 x_2 \\ 2\Gamma_2 y_2 \end{pmatrix} \quad \vec{\lambda_E} = \begin{pmatrix} \Gamma_1 \dot{y}_1 \\ -\Gamma_1 \dot{x}_1 \\ \Gamma_2 \dot{y}_2 \\ -\Gamma_2 \dot{x}_2 \end{pmatrix} \quad \vec{\lambda_X} = \begin{pmatrix} \Gamma_1 \\ 0 \\ \Gamma_2 \\ 0 \end{pmatrix} \quad \vec{\lambda_Y} = \begin{pmatrix} 0 \\ \Gamma_1 \\ 0 \\ \Gamma_2 \end{pmatrix}.$$

Each multiplier has a conserved quantity associated with it. To gather the conserved quantity Φ the product $\vec{\lambda} \cdot \vec{F}$ will be evaluated . This results in,

$$\vec{\lambda_L} \cdot \vec{F} = \Phi_L = \frac{d}{dt}\left(\Gamma_1(x_1^2 + y_1^2) + \Gamma_2(x_2^2 + y_2^2)\right),$$

$$\vec{\lambda_X} \cdot \vec{F} = \Phi_L = \frac{d}{dt}\left(\Gamma_1 x_1 + \Gamma_2 x_2\right),$$

$$\vec{\lambda_Y} \cdot \vec{F} = \Phi_L = \frac{d}{dt}\left(\Gamma_1 y_1 + \Gamma_2 y_2\right),$$

$$\vec{\lambda_E} \cdot \vec{F} = \Phi_L = \frac{d}{dt}\left(\frac{1}{4\pi}\Gamma_1\Gamma_2 \ln|\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2|}\right).$$

It is worthwhile to mention that the conserved quantitiesΦ_x, Φ_y, Φ_L and Φ_E are x-component of momentum, y-component of momentum, angular momentum, and Hamiltonian(Kinetic Energy) respectively. It was mentioned before that one conserved quantity may have more than one multipliers associated with it.There exists another multiplier$\vec{\lambda}_H$ which yields the Hamiltonian(H). In order to obtain $\vec{\lambda}_H$ one must write the system($\vec{F} = 0$) in Hamiltonian formalism such that,

$$\vec{F} = \begin{pmatrix} \dot{x}_1 + \frac{\partial H}{\partial y_1} \\ \dot{y}_1 - \frac{\partial H}{\partial x_1} \\ \dot{x}_2 + \frac{\partial H}{\partial y_2} \\ \dot{y}_2 - \frac{\partial H}{\partial x_2} \end{pmatrix}.$$

Choosing $\vec{\lambda}_H$ such that,

$$\begin{pmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial y_1} \\ \frac{\partial H}{\partial x_2} \\ \frac{\partial H}{\partial y_2} \end{pmatrix}.$$

Hence evaluating $\vec{\lambda}_H \cdot \vec{F}$ produces,

$$\dot{x}_1 \frac{\partial H}{\partial x_1} + \frac{\partial H}{\partial y_1}\frac{\partial H}{\partial x_1} + \dot{y}_1 \frac{\partial H}{\partial y_1} - \frac{\partial H}{\partial x_1}\frac{\partial H}{\partial y_1} + \dot{x}_2 \frac{\partial H}{\partial x_2} - \frac{\partial H}{\partial y_2}\frac{\partial H}{\partial x_2} + \dot{y}_2 \frac{\partial H}{\partial y_2} + \frac{\partial H}{\partial x_2}\frac{\partial H}{\partial y_2} = \frac{d}{dt}(H) = \frac{d}{dt}\left(\frac{1}{4\pi}\Gamma_1\Gamma_2 \ln|\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}|\right).$$

It is known in the literature that there are four independent constants of motion for point vortex problem.We found the corresponding multipliers for two point vortices and generalized to N-vortices.

Conservation of Linear Momentum in X- axis:

$$\vec{\lambda_X} = \begin{pmatrix} \Gamma_1 \\ 0 \\ \Gamma_2 \\ 0 \\ \Gamma_3 \\ . \\ . \\ . \\ \Gamma_N \\ 0 \end{pmatrix} \qquad \vec{\lambda_X} \cdot \vec{F} = \frac{d}{dt}(\sum_{i=1}^{N} \Gamma_i X_i)$$

Conservation of Linear Momentum in Y- axis:

$$\vec{\lambda_Y} = \begin{pmatrix} 0 \\ \Gamma_1 \\ 0 \\ \Gamma_2 \\ 0 \\ \Gamma_3 \\ . \\ . \\ . \\ 0 \\ \Gamma_N \end{pmatrix} \qquad \vec{\lambda_Y} \cdot \vec{F} = \frac{d}{dt}(\sum_{i=1}^{N} \Gamma_i Y_i)$$

Conservation of Angular Momentum:

$$\vec{\lambda_L} = \begin{pmatrix} 2\Gamma_1 X_1 \\ 2\Gamma_1 Y_1 \\ 2\Gamma_2 X_2 \\ 2\Gamma_2 Y_2 \\ 2\Gamma_3 X_3 \\ 2\Gamma_3 Y_3 \\ . \\ . \\ . \\ 2\Gamma_N X_N \\ 2\Gamma_N Y_N \end{pmatrix} \qquad \vec{\lambda_L} \cdot \vec{F} = \frac{d}{dt}(\sum_{i=1}^{N} \Gamma_i (X_i^2 + Y_i^2))$$

Conservation of Energy (Hamiltonian):

$$\vec{\lambda_H} = \begin{pmatrix} \frac{1}{2\pi}\Gamma_1 \sum_{i=2}^{N} \Gamma_i \frac{x_1 - x_i}{r_{1,i}^2} \\ \frac{1}{2\pi}\Gamma_1 \sum_{i=2}^{N} \Gamma_i \frac{y_1 - y_i}{r_{1,i}^2} \\ \frac{1}{2\pi}\Gamma_2 \sum_{\substack{i=1 \\ i\neq 2}}^{N} \Gamma_i \frac{x_2 - x_i}{r_{2,i}^2} \\ \frac{1}{2\pi}\Gamma_2 \sum_{\substack{i=1 \\ i\neq 2}}^{N} \Gamma_i \frac{y_2 - y_i}{r_{2,i}^2} \\ \frac{1}{2\pi}\Gamma_3 \sum_{\substack{i=1 \\ i\neq 3}}^{N} \Gamma_i \frac{x_3 - x_i}{r_{3,i}^2} \\ \frac{1}{2\pi}\Gamma_3 \sum_{\substack{i=1 \\ i\neq 3}}^{N} \Gamma_i \frac{y_3 - y_i}{r_{3,i}^2} \\ . \\ . \\ . \\ \frac{1}{2\pi}\Gamma_N \sum_{\substack{i=1 \\ i\neq N}}^{N} \Gamma_i \frac{x_N - x_i}{r_{N,i}^2} \\ \frac{1}{2\pi}\Gamma_N \sum_{\substack{i=1 \\ i\neq N}}^{N} \Gamma_i \frac{y_N - y_i}{r_{N,i}^2} \end{pmatrix} \qquad \vec{\lambda_H} \cdot \vec{F} = \frac{1}{4\pi}\frac{d}{dt}(\sum_{i,j=1}^{N} \Gamma_i \Gamma j \ln(r_{i,j}))$$

Multipliers can be collected in to one matrix Λ as row vectors and conservation laws can be collected in to one column vector Φ such that ,

$$\Lambda = \begin{pmatrix} \Gamma_1 & 0 & \dots & \Gamma_N & 0 \\ 0 & \Gamma_1 & \dots & 0 & \Gamma_N \\ 2\Gamma_1 X_1 & 2\Gamma_1 Y_1 & \dots & 2\Gamma_N X_N & 2\Gamma_N Y_N \\ \frac{1}{2\pi}\Gamma_1 \sum_{i=2}^{N} \Gamma_i \frac{x_1 - x_i}{r_{1,i}^2} & \frac{1}{2\pi}\Gamma_1 \sum_{i=2}^{N} \Gamma_i \frac{y_1 - y_i}{r_{1,i}^2} & \dots & \frac{1}{2\pi}\Gamma_N \sum_{\substack{i=1 \\ i\neq N}}^{N} \Gamma_i \frac{x_N - x_i}{r_{N,i}^2} & \frac{1}{2\pi}\Gamma_N \sum_{\substack{i=1 \\ i\neq N}}^{N} \Gamma_i \frac{y_N - y_i}{r_{N,i}^2} \end{pmatrix},$$

$$\vec{\Phi} = \begin{pmatrix} \sum_{i=1}^{N} \Gamma_i X_i \\ \sum_{i=1}^{N} \Gamma_i Y_i \\ \sum_{i=1}^{N} \Gamma_i (X_i^2 + Y_i^2) \\ \frac{1}{4\pi} \sum_{i,j=1}^{N} \Gamma_i \Gamma j \ln(r_{i,j}) \end{pmatrix}.$$

Chapter 3

Multiplier Method

In this section, the multiplier and the conserved quantity will be discretized in order to obtain a conservative method for the point vortex problem in the case of $N = 2, N = 3$ vortices.

3.1 Multiplier method for ODEs

We review the multiplier method for ODEs appeared in [3]. It was mentioned before that

$$\Lambda F = D_t(\phi),$$

where Λ is the matrix that combines the multipliers as row vectors, F is the vector that represents the equation, and $D_t(\phi)$ is the total derivative of the conserved quantity ϕ with respect to time. If Λ is a square matrix, isolating F will result in,

$$F = \Lambda^{-1} D_t(\phi).$$

However, if the multiplier matrix is rectangular, which is usually the case, one must partition F and Λ into two matrices as follows,

$$\begin{pmatrix} \tilde{\Lambda} & \Sigma \end{pmatrix} \begin{pmatrix} \tilde{F} \\ G \end{pmatrix} = D_t(\phi).$$

Isolating $\tilde{F}$ leads to,

$$\tilde{F} = \tilde{\Lambda}^{-1}(D_t(\phi) - \Sigma G). \tag{3.1}$$

Discretization will be symbolized with the superscript τ. Hence the discretized version of [3.1] becomes,

$$\tilde{F}^\tau = (\tilde{\Lambda}^\tau)^{-1}(D_t(\phi)^\tau - \Sigma^\tau G^\tau).$$

Thus the discretization of F will depend on the discrete determinant of $\tilde{\Lambda^\tau}$. This determinant shall not vanish at the mesh points in order for numerical solution to be robust. One can find many discretizations for $\tilde{F}$ by choosing various discretizations for $D_t(\phi), \Sigma, G$ and $\tilde{\Lambda}^{-1}$. By a smart choice of discretization for D_t and G^τ the discrete determinant $\tilde{\Lambda^\tau}$ can get factored out.

For $N = 2$, Conserving Linear Momentum in x, y gives,

$$(\tilde{\Lambda}^\tau)^{-1} = \frac{1}{\Gamma_1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{\Sigma}^\tau = \begin{pmatrix} \Gamma_2 & 0 \\ 0 & \Gamma_2 \end{pmatrix},$$

$$D_t(\phi)^\tau = \begin{pmatrix} \Gamma_1\left(\frac{x_1^{k+1}-x_1^k}{\tau}\right) + \Gamma_2\left(\frac{x_2^{k+1}-x_2^k}{\tau}\right) \\ \Gamma_1\left(\frac{y_1^{k+1}-y_1^k}{\tau}\right) + \Gamma_2\left(\frac{y_2^{k+1}-y_2^k}{\tau}\right) \end{pmatrix},$$

$$G^\tau = \begin{pmatrix} \frac{x_2^{k+1}-x_2^k}{\tau} - \frac{1}{2\pi}\frac{\Gamma_1(y_1^k-y_2^k)}{(x_2^k-x_1^k)^2+(y_2^k-y_1^k)^2} \\ \frac{y_2^{k+1}-y_2^k}{\tau} + \frac{1}{2\pi}\frac{\Gamma_1(x_1^k-x_2^k)}{(x_2^k-x_1^k)^2+(y_2^k-y_1^k)^2} \end{pmatrix},$$

$$\tilde{F}^\tau = \begin{pmatrix} \frac{x_1^{k+1}-x_1^k}{\tau} + \frac{1}{2\pi}\frac{\Gamma_2(y_1^k-y_2^k)}{(x_2^k-x_1^k)^2+(y_2^k-y_1^k)^2} \\ \frac{y_1^{k+1}-y_1^k}{\tau} - \frac{1}{2\pi}\frac{\Gamma_2(x_1^k-x_2^k)}{(x_2^k-x_1^k)^2+(y_2^k-y_1 k)^2} \end{pmatrix}.$$

For this case the determinant of $\tilde{\Lambda}^\tau$ was a constant hence did not vanish. Forward Euler-method has been chosen as discretization scheme.

$N = 2$, Conserving Angular Momentum gives,

$$(\tilde{\Lambda}^\tau)^{-1} = \begin{pmatrix} \Gamma_1\frac{1}{(x_1^{k+1}+x_1^k)} \end{pmatrix}, \qquad \tilde{\Sigma}^\tau = \begin{pmatrix} \Gamma_1(y_1^{k+1}+y_1^k) & \Gamma_2(x_1^{k+1}+x_1^k) & \Gamma_2(y_2^{k+1}+y_2^k) \end{pmatrix},$$

$$D_t(\phi)^\tau = \begin{pmatrix} \Gamma_1\left(\frac{(x_1^{k+1})^2-(x_1^k)^2}{\tau}\right) + \Gamma_1\left(\frac{(y_1^{k+1})^2-(y_1^k)^2}{\tau}\right) + \Gamma_2\left(\frac{(x_2^{k+1})^2-(x_2^k)^2}{\tau}\right) + \Gamma_2\left(\frac{(y_2^{k+1})^2-(y_2^k)^2}{\tau}\right) \end{pmatrix},$$

$$G^\tau = \begin{pmatrix} \frac{y_1^{k+1}-y_1^k}{\tau} - \frac{1}{4\pi}\frac{\Gamma_2(x_1^{k+1}-x_2^{k+1}+x_1^k-x_2^k)}{(x_2^{k+1}-x_1^{k+1})^2+(y_2^{k+1}-y_1^{k+1})^2} \\ \frac{x_2^{k+1}-x_2^k}{\tau} - \frac{1}{4\pi}\frac{\Gamma_1(y_1^{k+1}-y_2^{k+1}+y_1^k-y_2^k)}{(x_2^{k+1}-x_1^{k+1})^2+(y_2^{k+1}-y_1^{k+1})^2} \\ \frac{y_2^{k+1}-y_2^k}{\tau} + \frac{1}{4\pi}\frac{\Gamma_1(x_1^{k+1}-x_2^{k+1}+x_1^k-x_2^k)}{(x_2^{k+1}-x_1^{k+1})^2+(y_2^{k+1}-y_1^{k+1})^2} \end{pmatrix},$$

$$\tilde{F}^\tau = \begin{pmatrix} \frac{x_1^{k+1}-x_1^k}{\tau} + \frac{1}{4\pi}\frac{\Gamma_2(y_1^{k+1}-y_2^{k+1}+y_1^k-y_2^k)}{(x_2^{k+1}-x_1^{k+1})^2+(y_2^{k+1}-y_1^{k+1})^2} \end{pmatrix}. \tag{3.2}$$

This discretization of G^τ has been chosen on purpose in order to factor out the determinant $\tilde{\Lambda}^\tau$. If had used backward-Euler scheme for G^τ we would have obtained,

$$(\tilde{\Lambda}^\tau)^{-1} = \begin{pmatrix} \frac{1}{\Gamma_1 x_1^k} \end{pmatrix}, \qquad \tilde{\Sigma}^\tau = \begin{pmatrix} \Gamma_1(y_1^k) & \Gamma_2(x_1^k) & \Gamma_2(y_2^k) \end{pmatrix},$$

$$D_t(\phi)^\tau = \begin{pmatrix} \Gamma_1\left(\frac{(x_1^{k+1})^2-(x_1^k)^2}{\tau}\right) + \Gamma_1\left(\frac{(y_1^{k+1})^2-(y_1^k)^2}{\tau}\right) + \Gamma_2\left(\frac{(x_2^{k+1})^2-(x_2^k)^2}{\tau}\right) + \Gamma_2\left(\frac{(y_2^{k+1})^2-(y_2^k)^2}{\tau}\right) \end{pmatrix},$$

$$G^\tau = \begin{pmatrix} \frac{y_1^{k+1}-y_1^k}{\tau} - \frac{1}{2\pi}\frac{\Gamma_2(x_1^{k+1}-x_2^{k+1})}{(x_2^{k+1}-x_1^{k+1})^2+(y_2^{k+1}-y_1^{k+1})^2} \\ \frac{x_2^{k+1}-x_2^k}{\tau} - \frac{1}{2\pi}\frac{\Gamma_1(y_1^{k+1}-y_2^{k+1})}{(x_2^{k+1}-x_1^{k+1})^2+(y_2^{k+1}-y_1^{k+1})^2} \\ \frac{y_2^{k+1}-y_2^k}{\tau} + \frac{1}{2\pi}\frac{\Gamma_1(x_1^{k+1}-x_2^{k+1})}{(x_2^{k+1}-x_1^{k+1})^2+(y_2^{k+1}-y_1^{k+1})^2} \end{pmatrix},$$

$$\tilde{F}^\tau = \left(\frac{x_1^{k+1}-x_1^k}{x_1^k}\right)\begin{pmatrix} \frac{x_1^{k+1}-x_1^k}{\tau} + \frac{1}{2\pi}\frac{\Gamma_2(y_1^{k+1}-y_2^{k+1})}{(x_2^{k+1}-x_1^{k+1})^2+(y_2^{k+1}-y_1^{k+1})^2} \end{pmatrix}. \tag{3.3}$$

It can be seen from [3.3] that $\tilde{F}^\tau$ vanishes if the value of x_1^k is zero at one of the mesh points.

$N = 2$, Conserving Linear Momentum in x,y and Angular Momentum gives,

$$(\tilde{\Lambda}^\tau)^{-1} = \frac{1}{(\Gamma_1^2\Gamma_2)(x_2^{k+1}+x_2^k-x_1^{k+1}-x_1^k)}\begin{pmatrix} \Gamma_1\Gamma_2(x_2^{k+1}+x_2^k) & \Gamma_1\Gamma_2(y_1^{k+1}+y_1^k) & -\Gamma_1\Gamma_2 \\ 0 & \Gamma_1\Gamma_2(x_2^{k+1}+x_2^k-x_1^{k+1}-x_1^k) & 0 \\ -\Gamma_1^2(x_1^{k+1}+x_1^k) & -\Gamma_1^2(y_1^{k+1}+y_1^k) & -\Gamma_1^2 \end{pmatrix},$$

$$\tilde{\Sigma}^\tau = \begin{pmatrix} 0 \\ \Gamma_2 \\ \Gamma_2(y_2^{k+1}+y_2^k) \end{pmatrix},$$

$$D_t(\phi)^\tau = \begin{pmatrix} \Gamma_1\left(\frac{x_1^{k+1}-x_1^k}{\tau}\right)+\Gamma_2\left(\frac{x_2^{k+1}-x_2^k}{\tau}\right) \\ \Gamma_1\left(\frac{y_1^{k+1}-y_1^k}{\tau}\right)+\Gamma_2\left(\frac{y_2^{k+1}-y_2^k}{\tau}\right) \\ \Gamma_1\left(\frac{(x_1^{k+1})^2-(x_1^k)^2}{\tau}\right)+\Gamma_1\left(\frac{(y_1^{k+1})^2-(y_1^k)^2}{\tau}\right)+\Gamma_2\left(\frac{(x_2^{k+1})^2-(x_2^k)^2}{\tau}\right)+\Gamma_2\left(\frac{(y_2^{k+1})^2-(y_2^k)^2}{\tau}\right) \end{pmatrix},$$

$$G^\tau = \begin{pmatrix} \frac{y_2^{k+1}-y_2^k}{\tau} + \frac{1}{4\pi}\frac{\Gamma_1(x_1^{k+1}-x_2^{k+1}+x_1^k-x_2^k)}{(x_2^k-x_1^k)^2+(y_2^k-y_1^k)^2} \end{pmatrix},$$

$$\tilde{F}^\tau = \begin{pmatrix} \frac{x_1^{k+1}-x_1^k}{\tau} + \frac{1}{4\pi}\frac{\Gamma_2(y_1^{k+1}-y_2^{k+1}+y_1^k-y_2^k)}{(x_2^k-x_1^k)^2+(y_2^k-y_1^k)^2} \\ \frac{y_1^{k+1}-y_1^k}{\tau} - \frac{1}{4\pi}\frac{\Gamma_2(x_1^{k+1}-x_2^{k+1}+x_1^k-x_2^k)}{(x_2^k-x_1^k)^2+(y_2^k-y_1^k)^2} \\ \frac{x_2^{k+1}-x_2^k}{\tau} - \frac{1}{4\pi}\frac{\Gamma_1(y_1^{k+1}-y_2^{k+1}+y_1^k-y_2^k)}{(x_2^k-x_1^k)^2+(y_2^k-y_1^k)^2} \end{pmatrix}.$$

For $N=3$, Conserving Linear Momentum in x,y gives,

$$(\tilde{\Lambda}^\tau)^{-1} = \frac{1}{\Gamma_1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{\Sigma}^\tau = \begin{pmatrix} \Gamma_2 & 0 & \Gamma_3 & 0 \\ 0 & \Gamma_2 & 0 & \Gamma_3 \end{pmatrix},$$

$$D_t(\phi)^\tau = \begin{pmatrix} \Gamma_1\left(\frac{x_1^{k+1}-x_1^k}{\tau}\right)+\Gamma_2\left(\frac{x_2^{k+1}-x_2^k}{\tau}\right)+\Gamma_3\left(\frac{x_3^{k+1}-x_3^k}{\tau}\right) \\ \Gamma_1\left(\frac{y_1^{k+1}-y_1^k}{\tau}\right)+\Gamma_2\left(\frac{y_2^{k+1}-y_2^k}{\tau}\right)+\Gamma_3\left(\frac{y_2^{k+1}-y_2^k}{\tau}\right) \end{pmatrix},$$

$$G^\tau = \begin{pmatrix} \frac{x_2^{k+1}-x_2^k}{\tau} + \frac{1}{2\pi}\frac{\Gamma_1(y_2^k-y_1^k)}{(x_2^k-x_1^k)^2+(y_2^k-y_1^k)^2} + \frac{1}{2\pi}\frac{\Gamma_3(y_2^k-y_3^k)}{(x_3^k-x_2^k)^2+(y_3^k-y_2^k)^2} \\ \frac{y_2^{k+1}-y_2^k}{\tau} - \frac{1}{2\pi}\frac{\Gamma_1(x_2^k-x_1^k)}{(x_2^k-x_1^k)^2+(y_2^k-y_1^k)^2} - \frac{1}{2\pi}\frac{\Gamma_3(x_2^k-x_3^k)}{(x_3^k-x_2^k)^2+(y_3^k-y_2^k)^2} \\ \frac{x_3^{k+1}-x_3^k}{\tau} + \frac{1}{2\pi}\frac{\Gamma_1(y_3^k-y_1^k)}{(x_3^k-x_1^k)^2+(y_3^k-y_1^k)^2} + \frac{1}{2\pi}\frac{\Gamma_2(y_3^k-y_2^k)}{(x_3^k-x_2^k)^2+(y_3^k-y_2^k)^2} \\ \frac{y_3^{k+1}-y_3^k}{\tau} - \frac{1}{2\pi}\frac{\Gamma_1(x_3^k-x_1^k)}{(x_3^k-x_1^k)^2+(y_3^k-y_1^k)^2} - \frac{1}{2\pi}\frac{\Gamma_2(x_3^k-x_2^k)}{(x_3^k-x_2^k)^2+(y_3^k-y_2^k)^2} \end{pmatrix},$$

$$\tilde{F}^\tau = \begin{pmatrix} \frac{x_1^{k+1}-x_1^k}{\tau} + \frac{1}{2\pi}\frac{\Gamma_2(y_1^k-y_2^k)}{(x_2^k-x_1^k)^2+(y_2^k-y_1^k)^2} + \frac{1}{2\pi}\frac{\Gamma_3(y_1^k-y_3^k)}{(x_3^k-x_1^k)^2+(y_3^k-y_1^k)^2} \\ \frac{y_1^{k+1}-y_1^k}{\tau} - \frac{1}{2\pi}\frac{\Gamma_2(x_1^k-x_2^k)}{(x_2^k-x_1^k)^2+(y_2^k-y_1k)^2} - \frac{1}{2\pi}\frac{\Gamma_3(x_1^k-x_3^k)}{(x_3^k-x_1^k)^2+(y_3^k-y_1^k)^2} \end{pmatrix}.$$

$N=3$, Conserving Angular Momentum gives,

$$(\tilde{\Lambda}^\tau)^{-1} = \begin{pmatrix} \Gamma_1\frac{1}{(x_1^{k+1}+x_1^k)} \end{pmatrix}, \qquad \tilde{\Sigma}^\tau = \begin{pmatrix} \Gamma_1(y_1^{k+1}+y_1^k) & \Gamma_2(x_1^{k+1}+x_1^k) & \dots & \Gamma_3(y_3^{k+1}+y_3^k) \end{pmatrix},$$

$$D_t(\phi)^\tau = \begin{pmatrix} \Gamma_1\left(\frac{(x_1^{k+1})^2-(x_1^k)^2}{\tau}\right)+\Gamma_1\left(\frac{(y_1^{k+1})^2-(y_1^k)^2}{\tau}\right)+\ldots+\Gamma_3\left(\frac{(x_3^{k+1})^2-(x_3^k)^2}{\tau}\right)+\Gamma_3\left(\frac{(y_3^{k+1})^2-(y_3^k)^2}{\tau}\right) \end{pmatrix},$$

$$
G^\tau = \begin{pmatrix}
\frac{y_1^{k+1}-y_1^k}{\tau} - \frac{1}{4\pi}\frac{\Gamma_2(x_1^{k+1}-x_2^{k+1}+x_1^k-x_2^k)}{(x_2^k-x_1^{k+1})^2+(y_2^k-y_1^{k+1})^2} - \frac{1}{4\pi}\frac{\Gamma_3(x_1^{k+1}-x_3^{k+1}+x_1^k-x_3^k)}{(x_3^k-x_1^k)^2+(y_3^k-y_1^k)^2} \\
\frac{x_2^{k+1}-x_2^k}{\tau} + \frac{1}{4\pi}\frac{\Gamma_1(y_2^{k+1}-y_1^{k+1}+y_2^k-y_1^k)}{(x_2^k-x_1^k)^2+(y_2^k-y_1^k)^2} + \frac{1}{4\pi}\frac{\Gamma_3(y_2^{k+1}-y_3^{k+1}+y_2^k-y_3^k)}{(x_2^k-x_3^k)^2+(y_2^k-y_3^k)^2} \\
\frac{y_2^{k+1}-y_2^k}{\tau} - \frac{1}{4\pi}\frac{\Gamma_1(x_2^{k+1}-x_1^{k+1}+x_2^k-x_1^k)}{(x_2^k-x_1^k)^2+(y_2^k-y_1^k)^2} - \frac{1}{4\pi}\frac{\Gamma_3(x_2^{k+1}-x_3^{k+1}+x_2^k-x_3^k)}{(x_2^k-x_3^k)^2+(y_2^k-y_3^k)^2} \\
\frac{x_3^{k+1}-x_3^k}{\tau} + \frac{1}{4\pi}\frac{\Gamma_1(y_3^{k+1}-y_1^{k+1}+y_3^k-y_1^k)}{(x_3^k-x_1^k)^2+(y_3^k-y_1^k)^2} + \frac{1}{4\pi}\frac{\Gamma_2(y_3^{k+1}-y_2^{k+1}+y_3^k-y_2^k)}{(x_2^k-x_3^k)^2+(y_2^k-y_3^k)^2} \\
\frac{y_3^{k+1}-y_3^k}{\tau} - \frac{1}{4\pi}\frac{\Gamma_1(x_3^{k+1}-x_1^{k+1}+x_3^k-x_1^k)}{(x_3^k-x_1^k)^2+(y_3^k-y_1^k)^2} - \frac{1}{4\pi}\frac{\Gamma_2(x_3^{k+1}-x_2^{k+1}+x_3^k-x_2^k)}{(x_3^k-x_2^k)^2+(y_3^k-y_2^k)^2}
\end{pmatrix},
$$

$$
\tilde{F}^\tau = \begin{pmatrix} \frac{x_1^{k+1}-x_1^k}{\tau} + \frac{1}{4\pi}\frac{\Gamma_2(y_1^{k+1}-y_2^{k+1}+x_1^k-x_2^k)}{(x_2^k-x_1^k)^2+(y_2^k-y_1^k)^2} + \frac{1}{4\pi}\frac{\Gamma_3(y_1^{k+1}-y_3^{k+1}+x_1^k-x_3^k)}{(x_3^k-x_1^k)^2+(y_3^k-y_1^k)^2} \end{pmatrix}.
$$

$N = 3$, Conserving Linear Momentum in x,y and Angular Momentum gives,

$$
(\tilde{\Lambda}^\tau)^{-1} = \frac{1}{(\Gamma_1^2\Gamma_2)(x_2^{k+1}+x_2^k-x_1^{k+1}-x_1^k)} \begin{pmatrix}
\Gamma_1\Gamma_2(x_2^{k+1}+x_2^k) & \Gamma_1\Gamma_2(y_1^{k+1}+y_1^k) & -\Gamma_1\Gamma_2 \\
0 & \Gamma_1\Gamma_2(x_2^{k+1}+x_2^k-x_1^{k+1}-x_1^k) & 0 \\
-\Gamma_1^2(x_1^{k+1}+x_1^k) & -\Gamma_1^2(y_1^{k+1}+y_1^k) & -\Gamma_1^2
\end{pmatrix},
$$

$$
\tilde{\Sigma}^\tau = \begin{pmatrix}
0 & \Gamma_3 & 0 \\
\Gamma_2 & 0 & \Gamma_3 \\
\Gamma_2(y_2^{k+1}+y_2^k) & \Gamma_3(x_3^{k+1}+x_3^k) & \Gamma_3(y_3^{k+1}+y_3^k)
\end{pmatrix},
$$

$$
D_t(\phi)^\tau = \begin{pmatrix}
\Gamma_1\left(\frac{x_1^{k+1}-x_1^k}{\tau}\right) + \Gamma_2\left(\frac{x_2^{k+1}-x_2^k}{\tau}\right) + \Gamma_3\left(\frac{x_3^{k+1}-x_3^k}{\tau}\right) \\
\Gamma_1\left(\frac{y_1^{k+1}-y_1^k}{\tau}\right) + \Gamma_2\left(\frac{y_2^{k+1}-y_2^k}{\tau}\right) + \Gamma_3\left(\frac{y_2^{k+1}-y_2^k}{\tau}\right) \\
\Gamma_1\left(\frac{(x_1^{k+1})^2-(x_1^k)^2}{\tau}\right) + \Gamma_1\left(\frac{(y_1^{k+1})^2-(y_1^k)^2}{\tau}\right) \ldots + \Gamma_3\left(\frac{(x_3^{k+1})^2-(x_3^k)^2}{\tau}\right) + \Gamma_3\left(\frac{(y_3^{k+1})^2-(y_3^k)^2}{\tau}\right)
\end{pmatrix},
$$

$$
G^\tau = \begin{pmatrix}
\frac{y_2^{k+1}-y_2^k}{\tau} - \frac{1}{4\pi}\frac{\Gamma_1(x_2^{k+1}-x_1^{k+1}+x_2^k-x_1^k)}{(x_2^k-x_1^k)^2+(y_2^k-y_1^k)^2} - \frac{1}{4\pi}\frac{\Gamma_3(x_2^{k+1}-x_3^{k+1}+x_2^k-x_3^k)}{(x_2^k-x_3^k)^2+(y_2^k-y_3^k)^2} \\
\frac{x_3^{k+1}-x_3^k}{\tau} + \frac{1}{4\pi}\frac{\Gamma_1(y_3^{k+1}-y_1^{k+1}+y_3^k-y_1^k)}{(x_3^k-x_1^k)^2+(y_3^k-y_1^k)^2} + \frac{1}{4\pi}\frac{\Gamma_2(y_3^{k+1}-y_2^{k+1}+y_3^k-y_2^k)}{(x_2^k-x_3^k)^2+(y_2^k-y_3^k)^2} \\
\frac{y_3^{k+1}-y_3^k}{\tau} - \frac{1}{4\pi}\frac{\Gamma_1(x_3^{k+1}-x_1^{k+1}+x_3^k-x_1^k)}{(x_3^k-x_1^k)^2+(y_3^k-y_1^k)^2} - \frac{1}{4\pi}\frac{\Gamma_2(x_3^{k+1}-x_2^{k+1}+x_3^k-x_2^k)}{(x_3^k-x_2^k)^2+(y_3^k-y_2^k)^2}
\end{pmatrix},
$$

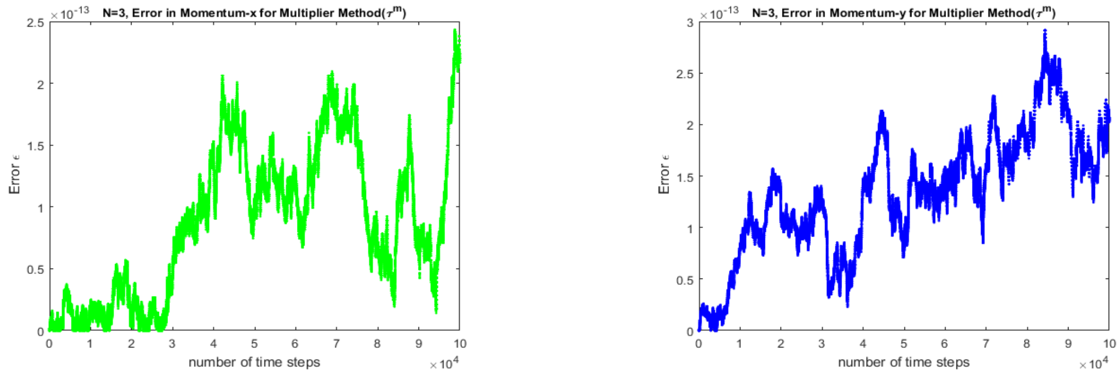
$$
\tilde{F}^\tau = \begin{pmatrix}
\frac{x_1^{k+1}-x_1^k}{\tau} + \frac{1}{4\pi}\frac{\Gamma_2(y_1^{k+1}-y_2^{k+1}+x_1^k-x_2^k)}{(x_2^k-x_1^k)^2+(y_2^k-y_1^k)^2} + \frac{1}{4\pi}\frac{\Gamma_3(y_1^{k+1}-y_3^{k+1}+x_1^k-x_3^k)}{(x_3^k-x_1^k)^2+(y_3^k-y_1^k)^2} \\
\frac{y_1^{k+1}-y_1^k}{\tau} - \frac{1}{4\pi}\frac{\Gamma_2(x_1^{k+1}-x_2^{k+1}+x_1^k-x_2^k)}{(x_2^k-x_1^{k+1})^2+(y_2^k-y_1^{k+1})^2} - \frac{1}{4\pi}\frac{\Gamma_3(x_1^{k+1}-x_3^{k+1}+x_1^k-x_3^k)}{(x_3^k-x_1^k)^2+(y_3^k-y_1^k)^2} \\
\frac{x_2^{k+1}-x_2^k}{\tau} + \frac{1}{4\pi}\frac{\Gamma_1(y_2^{k+1}-y_1^{k+1}+y_2^k-y_1^k)}{(x_2^k-x_1^k)^2+(y_2^k-y_1^k)^2} + \frac{1}{4\pi}\frac{\Gamma_3(y_2^{k+1}-y_3^{k+1}+y_2^k-y_3^k)}{(x_2^k-x_3^k)^2+(y_2^k-y_3^k)^2}
\end{pmatrix}.
$$

Chapter 4

Numerical Results

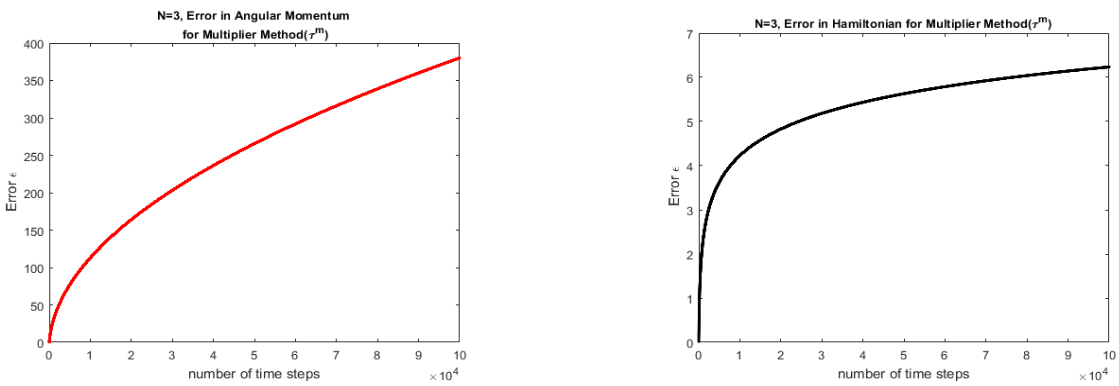
In this last section, we will present error plots of the conserved quantities for the conservative discretization from the previous section. We will compare the multiplier method with standard methods, particularly Runge-Kutta Methods.

4.1 Multiplier method



(a) (b)

Figure 4.1: $N = 3$, error in linear momentum in x,y using τ^m



(a) (b)

Figure 4.2: $N = 3$, error in angular momentum and Hamiltonian using τ^m

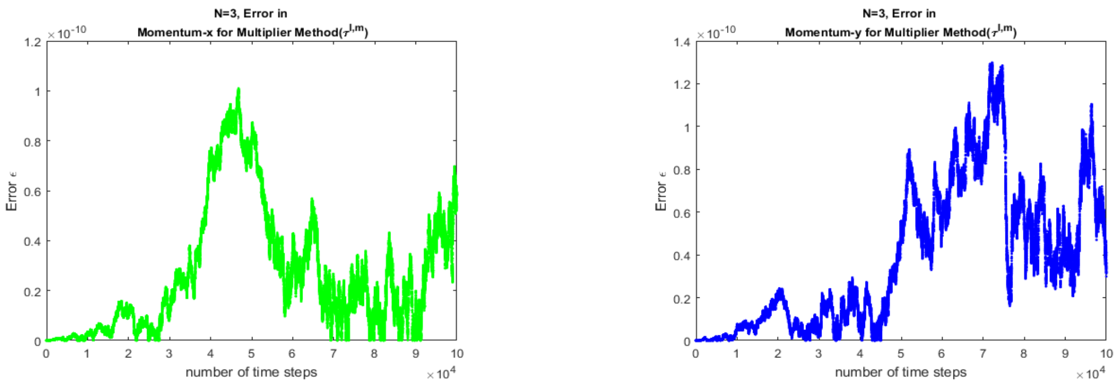
The error ϵ is defined in this context as,

$$|\Phi(\vec{x_0}) - \Phi(\vec{x_0} + n * \vec{dt})|,$$

where $\vec{x_0}$ is the vector of initial values given in the Table 4.1. The figures are constructed by generating ϵ for each n. It can be seen from [4.1a] and [4.1b] that the error in x and y component of the momentum is generated by machine round off. This verifies that the momenta-conserving discretization τ_m conserves momentum in x and y. However [4.2a] and [4.2b] show that angular momentum and the Hamiltonian are not conserved under this discretization. This is reasonable since τ_m was designed to only conserve momentum x and y.

Number of time steps	step size	initial value	Circulation strengths
100000	0.5	$x_1 = 1, y_1 = 2$	$\Gamma_1 = 1$
		$x_2 = 0, y_2 = -1$	$\Gamma_2 = 2$
		$x_3 = 2, x_3 = -1$	$\Gamma_3 = 3$

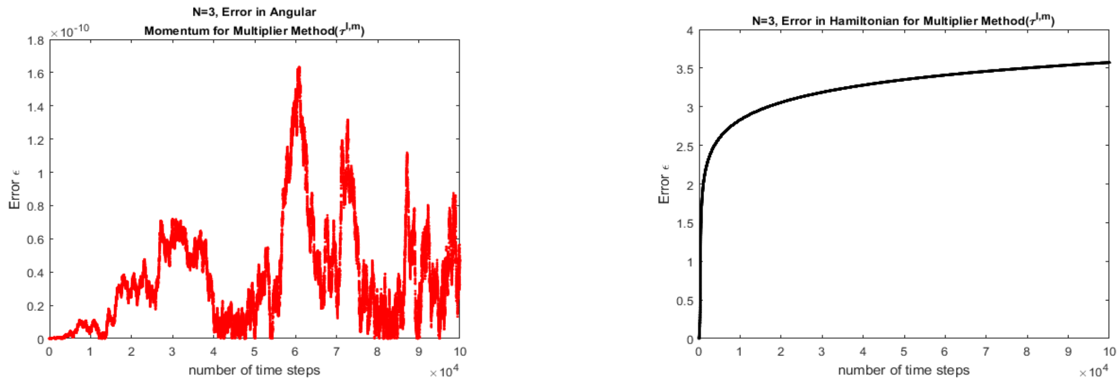
Table 4.1: Parameters



(a) (b)

Figure 4.3: $N = 3$, error linear momentum in x,y using $\tau^{l,m}$

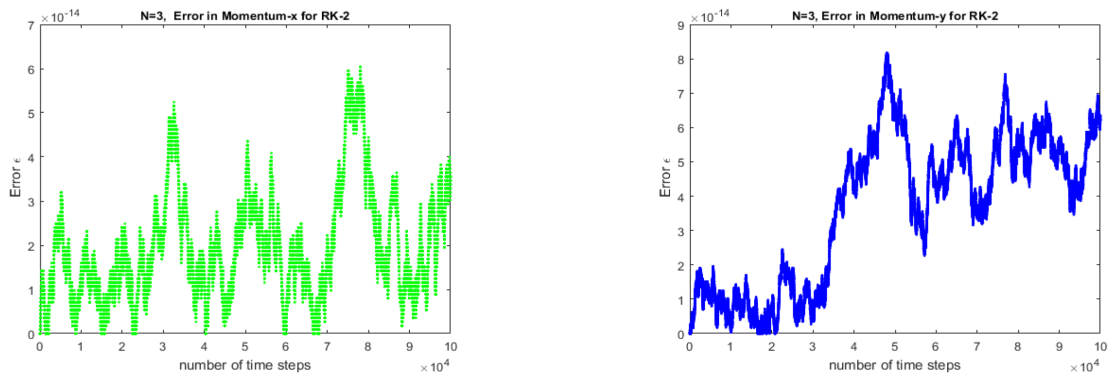
Figures [4.3a] , [4.3b] and [4.4a] show that momentum x,y and angular momentum are all conserved under the angular momentum and momenta-conserving discretization $\tau^{l,m}$.Comparing figures, [4.3a] and [4.3b] and figures [4.1a] and [4.1b] shows that the difference in the order of magnitude is a factor of three for τ_m and $\tau_{l,m}$. This is because the $\tau^{l,m}$ is an implicit discretization. Figure [4.4b] shows that $\tau_{l,m}$ does not conserve the Hamiltonian.



(a) (b)

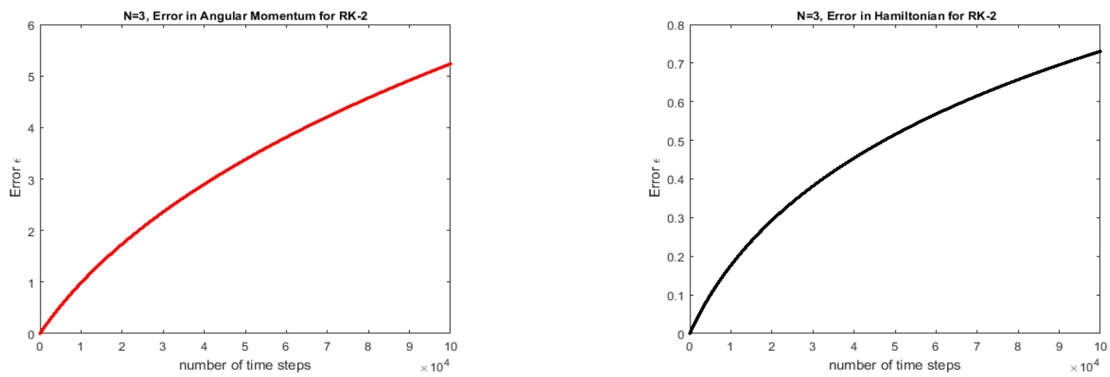
Figure 4.4: $N = 3$, error in angular momentum and Hamiltonian using $\tau^{l,m}$

4.2 Comparison with Runge-Kutta



(a) (b)

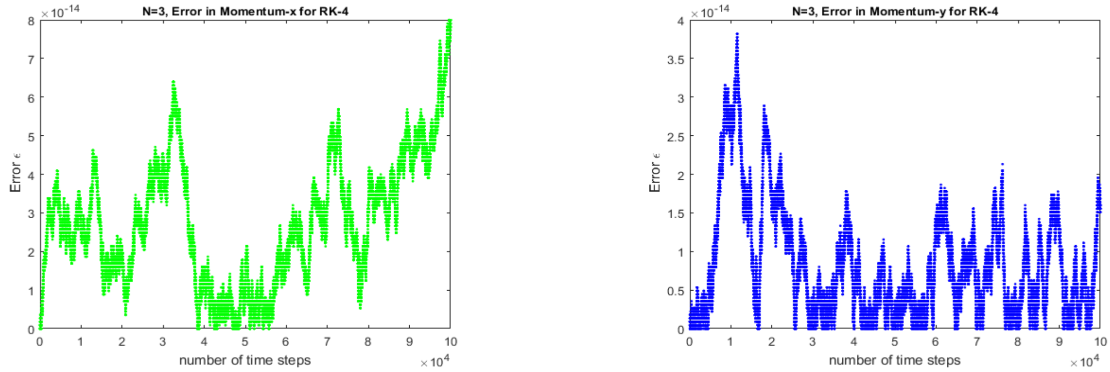
Figure 4.5: $N = 3$, error in linear momentum in x,y using RK-2



(a) (b)

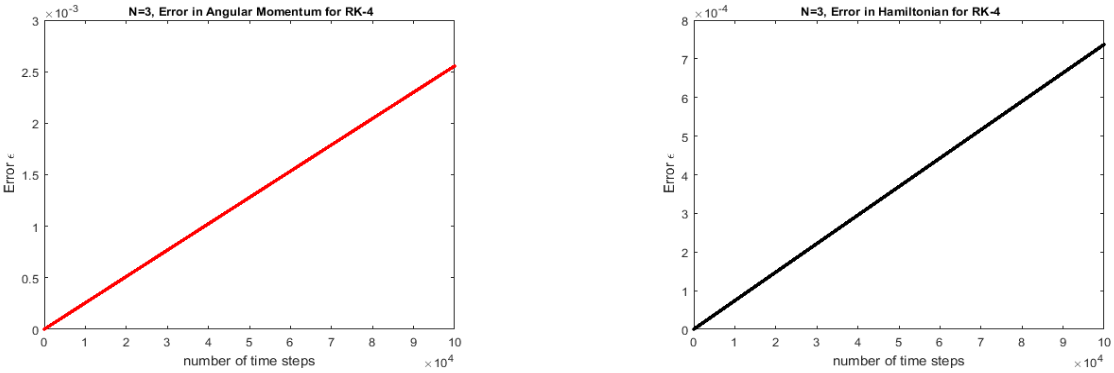
Figure 4.6: $N = 3$, error in angular momentum and Hamiltonian using RK-2

From [4.5a] and [4.5b] , it can be observed that RK-2 conserves the linear momenta, since the figures imply that the source of the error is mostly round off. On the contrary, RK-2 does not conserve the angular momentum and the Hamiltonian .



(a) (b)

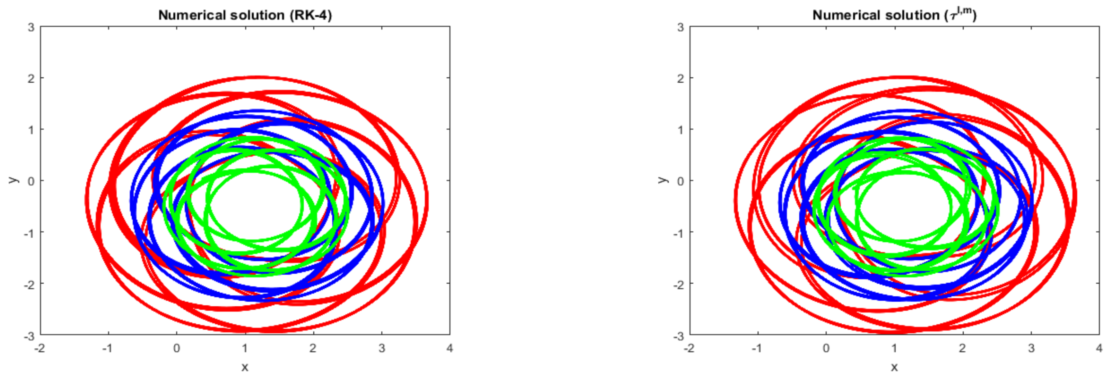
Figure 4.7: $N = 3$, error linear momentum in x,y using RK-4



(a) (b)

Figure 4.8: $N = 3$, error in angular momentum and Hamiltonian using RK-4

Similar to RK-2, RK-4 also conserves linear momenta without conserving angular momentum and Hamiltonian.



(a) (b)

Figure 4.9: $N = 3$, Path of vortices using RK-4 and $\tau^{l,m}$

The plots above were generated using parameters from Table 4.2. From figures [4.9a] and [4.9b] it can be seen that two figures are not exactly the same. RK-4 provides a more accurate solution then $\tau^{l,m}$ because it is an higher order method. However, higher order method does not mean more accurate solution in long time. Moreover, RK-4 is not able to preserve energy and angular momentum in the discrete level.

Number of time steps	step size	initial value	Circulation strengths
100000	0.005	$x_1 = 1, y_1 = 2$	$\Gamma_1 = 1$
		$x_2 = 0, y_2 = -1$	$\Gamma_2 = 2$
		$x_3 = 2, x_3 = -1$	$\Gamma_3 = 3$

Table 4.2: Parameters

Chapter 5

Conclusion

In this paper, conservation law multipliers and their corresponding conservation laws were derived for the 2-vortex problem and the N-vortex problem. Four conservation law multipliers were found for linear momenta, angular momentum and energy. The multipliers and the conserved quantities were discretized in order to obtain a conservative discretization for N=3. Two types of conservative discretizations were implemented: one which conserves only linear momenta and the other which conserves linear momenta and angular momentum. They were verified numerically to be conservative up to machine round-off. Moreover, they were compared to standard numerical schemes, such as RK-2 and RK-4.

There are three possible directions for future work: generalize the proposed discretization to N-vortices, find a conservative discretization which conserves all four conservation laws, and generalize the results to the point vortex problem on the sphere.

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[3] Andy T. S. Wan, Alexander Bihlo, and Jean-Christophe Nave. The Multiplier Method to Construct Conservative Finite Difference Schemes for Ordinary and Partial Differential Equations. *SIAM J. Numer. Anal.*, 54:86–119, 2016.