EIGENVECTORS OF ELEMENTS IN $\mathbb{Z}[\mathrm{SU}(2)]$

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1. Introduction

In the paper [GJS], the authors looked at eigenvalue distribution and level spacings distribution for the N-th irreducible representation of elements of the group ring of $SU(2)$ as $N \to \infty$.

In the current paper, we will study the behaviour of the corresponding *eigenvectors*.

2. Questions

Fix $k \geq 3$. Let $g = g_1 + g_1^{-1} + \ldots + g_k + g_k^{-1}$ be a generic element of $\mathbb{R}[\mathrm{SU}(2)]$. Let π_N denote the N-th irreducible representation of $\mathrm{SU}(2)$. It has been found in [GJS] that for N odd, the eigenvalues of $\pi_N(g)$ are *double* for generic choice of $g \in \mathrm{SU}(2)^k$, while for N even, the eigenvalues are *simple* for such g. We normalize eigenvectors to have l^2 norm equal to 1.

For N even, we get $(N+1)$ points on the unit sphere $S^N \subset \mathbb{R}^{N+1}$. It is a natural to ask the following:

Question 1. Do those points become uniformly distributed on S^N as $N \to \infty$?

A weaker question is whether this happens after averaging over $\mathrm{SU}(2)^k$.

For N odd, the eigenvalues are double. Accordingly, a natural question seems to be the following:

Question 2. Are the 2-dimensional subspaces corresponding to the double eigenvalues becoming uniformly distributed in $\mathrm{Gr}(2, N+1)$, as $N \to \infty$?

3. Results for random matrices

Our motivation comes from the theory of random matrices, where the behaviour of eigenvectors was studied by Gaudin, Mehta, Knowles, Yin, Tao and Vu. We refer to [KnY, TV] and references therein for results about both Wigner matrices and general random matrices. One of the important results is that the eigenvectors of random matrices become uniformly distributed on the unit sphere as the dimension of the matrix is growing.

4. Increasing the number of generators

In [GJS] the following result was shown ([GJS, Prop. 2.1]):

Proposition 4.1. *Let $\nu_{N,k}$ be the direct image of $dg_1 dg_2 \dots dg_k$ on $G^{(k)}$ under the map*

$$(g_1, g_2, \dots, g_k) \to (\frac{1}{\sqrt{k}}(g_1 + g_1^{-1} + \dots + g_k + g_k^{-1}))\widehat{\ }(\pi_N) \tag{4.1}$$

Thus $\nu_{N,k}$ is a probability measure on $\mathcal{H}_{N+1}$. As $k \to \infty$, $\nu_{N,k}$ converges in measure to the standard GOE measure on $\mathcal{H}_{N+1}$ if N is even and to the standard GSE measure if N is odd.

Here $\mathcal{H}_{N+1}$ denotes the real linear space of $(N+1)\times(N+1)$ real symmetric matrices.

Convergence of matrices implies convergence of their eigenvectors. Combining with results from [TV], we obtain the following results:

Theorem 4.2. *Fix N and let $k \to \infty$. Then the conclusions of* [TV, Thm. 3 i)] *and* [TV, Corollary 4] *holds for eigenvectors of* $(g_1 + g_1^{-1} + \dots + g_k + g_k^{-1}))\widehat{\ }(\pi_N)$.

In other words, this implies that as $k \to \infty$, after a respective normalization of the eigenvectors (discussed in [TV]), the coefficients of the eigenvectors (for $M = o(\sqrt{N})$ being a coefficient of the eigenvectors) of $\pi_N(g)$ (after scaling them by $\sqrt{N}$), will differ from an independent random variable by $o(1)$ in the variation norm. This result is further studied in the numerical results shown later in the paper. A weaker result holds if $M = o(N/\log N)$.

5. Numerical implementation

In this section, we *fix* the number k of generators of a subgroup of SU(2), and study numerically the distribution of the eigenvectors of 4.1.

The Lie algebra

$$\mathrm{su}(2) = \left\{ \begin{bmatrix} ia & z \\ -\overline{z} & -ia \end{bmatrix} : a \in \mathbb{R}, z \in \mathbb{C} \right\}$$

is spanned by

$$X_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \qquad X_2 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \qquad X_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

Together with the exponential map and the differential of π_N, for $X \in \mathrm{su}(2)$, the relation

$$\pi_N(\exp X) = \exp(d\pi_N(X)),$$

is used to compute π_N of some element of SU(2). First, we compute explicitely $d\pi_N$ of X_1, X_2 and X_3 with the relation $d\pi_N(X) = \frac{d}{dt}\exp(tX)\big|_{t=0}$. This allows us to find $d\pi_N$ of any element of su(2). Then, the matrix exponential of Mathematica or Matlab is used to the matrix in W_{N+1}. The remaining of the section are the computations of $d\pi_N$ for X_1, X_2, X_3.

Starting with X_1, we have $\exp tX_1 = \begin{bmatrix} e^{ti} & 0 \\ 0 & e^{-ti} \end{bmatrix}$. Since the action of $\exp tX_1$ sends $(x \quad y)$ to $(e^{it}x \quad e^{-it}y)$, with $e_j = x^j y^{N-j}$ being the basis of W_{N+1}, one gets, for $0 < j < N$,

$$\begin{aligned} \frac{d}{dt} \exp tX_1(e_j)\Big|_{t=0} &= \frac{d}{dt}(e^{it}x)^j(e^{-it}y)^{N-j}\Big|_{t=0} \\ &= \frac{d}{dt} e^{2itj-itN} x^j y^{N-j}\Big|_{t=0} \\ &= (2ij - iN)e_j \end{aligned}$$

and $-iNe_j$ for $j = 0$, iNe_j for $j = N$.

Now, we have the computation of $\exp(tX_2)$. We notice that (I_2 is the 2×2 identity matrix):

$$X_2^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -I_2 \quad X_2^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -X_2$$

$$X_2^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \quad X_2^5 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = X_2$$

This enables us to find:

$$\exp(tX_2) = I_2 + tX_2 - t^2 \frac{I_2}{2!} - t^3 \frac{X_2}{3!} + t^4 \frac{I_2}{4!} + t^5 \frac{X_5}{5!} + \dots$$

$$\exp(tX_2) = \begin{bmatrix} \cos(t) & 0 \\ 0 & \cos(t) \end{bmatrix} + \begin{bmatrix} 0 & \sin(t) \\ -\sin(t) & 0 \end{bmatrix} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

This is found by collecting terms and recognizing the infinite sums give the power series representation of $\sin(t)$ and $\cos(t)$.

Now, we have the following basis of W_{n+1} (for $j = 0 \dots, N$):

$$\hat{e}_j = \frac{x^j y^{N-j}}{\sqrt{j!(N-j)!}}$$

We will now compute $d\pi_N(X_2)(\hat{e_j})$. We claim that:

$$d\pi_N(X_2)(\hat{e_j}) = -\sqrt{j(N-j+1)}\hat{e}_{j-1} + \sqrt{(N-j)(j+1)}\hat{e}_{j+1}$$

Firstly, we have that the elements of the basis $\hat{e}_j$ are mapped to:

$$\hat{e}_j = \frac{x^j y^{N-j}}{\sqrt{j!(N-j)!}} \to \frac{(x\cos(t) - y\sin(t))^j (x\sin(t) + y\cos(t))^{N-j}}{\sqrt{j!(N-j)!}}$$

Now, it follows:

(5.1)
$$d\pi_N(X_2)(\hat{e}_j) = \frac{d}{dt}\left(\frac{(x\cos(t) - y\sin(t))^j (x\sin(t) + y\cos(t))^{N-j}}{\sqrt{j!(N-j)!}} \right)|_{t=0}$$

Assume now that $j \neq 0$ or $j \neq N$. Then, differentiating the above expression gives:

$$\begin{aligned} d\pi_N(X_2)(\hat{e}_j) = &\frac{1}{\sqrt{j!(N-j)!}} \big((-j(x\cos(t) - y\sin(t))^{j-1}(x\sin(t) + y\cos(t))^{N-j+1} \\ &+ \ (N-j)(x\sin(t) + y\cos(t))^{N-j-1}(x\cos(t) - y\sin(t))^{j+1}\big)|_{t=0} \end{aligned}$$

Evaluating the following expression, this reduces to:

$$d\pi_N(X_2)(\hat{e}_j) = \frac{-j}{\sqrt{j!(N-j)!}} x^{j-1}y^{N-(j-1)} + \frac{N-j}{\sqrt{j!(N-j)!}} y^{N-(j+1)}x^{j+1}$$

By rearranging the terms, the result follows for $j \neq 0$ or $j \neq N$:

$$d\pi_N(X_2)(\hat{e}_j) = -\sqrt{j(N-j+1)}\hat{e}_{j-1} + \sqrt{(N-j)(j+1)}\hat{e}_{j+1}$$

For $j = 0, N$, the computations are very similar, and we find $\sqrt{N}\widehat{e}_1$ and $-\sqrt{N}\widehat{e}_{N-1}$ respectively.

To now compute $d\pi_N(X_2)$, simply notice that the action of $d\pi_N(X_2)$ on the basis vectors $\hat{e}_j$ will give us the columns in $d\pi_N(X_2)$. Hence, it follows:

$$d\pi_N(X_2)(\hat{e}_j) = \begin{cases} \sqrt{N}\hat{e}_1 & \text{if } j = 0 \\ -\sqrt{j(N-j+1)}\hat{e}_{j-1} + \sqrt{(N-j)(j+1)}\hat{e}_{j+1} & \text{if } j \neq 0, j \neq N \\ -\sqrt{N}\hat{e}_{N-1} & \text{if } j = N \end{cases}$$

For X_3, we note that $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]^2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] = I_2$. Next, we have

$$\begin{aligned} \exp tX_3 &= \sum_{n=0}^{\infty} \frac{t^n {X_3}^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{t^n i^n}{n!} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^n \\ &= I \sum_{n=0}^{\infty} \frac{(-1)^{2n} t^{2n}}{(2n)!} + i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sum_{n=0}^{\infty} \frac{(-1)^{2n+1} t^{2n+1}}{(2n+1)!} \\ &= \begin{bmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{bmatrix}. \end{aligned}$$

We know the acion of $\exp tX_3$ $(x\ \ y) \mapsto (x\cos t + iy\sin t \quad ix\sin t + y\cos t)$. For e_j with $1 < j < N$, the computation becomes

$$\begin{aligned}
&\frac{d}{dt}(\exp tX_3)(e_j)\Bigg|_{t=0} \\
&\qquad = \frac{d}{dt}(x\cos t + iy\sin t)^j (ix\sin t + y\cos t)^{N-j}\Bigg|_{t=0} \\
&\qquad = j(x\cos t + iy\sin t)^{j-1}(-x\sin t + iy\cos t)(ix\sin t + y\cos t)^{N-j} \\
&\qquad\qquad + (x\cos t + iy\sin t)^j (N-j)(ix\sin t + y\cos t)^{N-j-1}(ix\cos t - y\cos t)\Bigg|_{t=0}
\end{aligned}$$

since $-x\sin t + iy\cos t = i(ix\sin t + y\cos t)$ and $i\cos t - y\sin t = i(x\cos t + iy\sin t)$,

$$\begin{aligned}
&\qquad = ij(x\cos t + iy\sin t)^{j-1}(ix\sin t + y\cos t)^{N-j+1} \\
&\qquad\qquad + i(N-j)(x\cos t + iy\sin t)^{j+1}(ix\sin t + y\cos t)^{N-j-1}\Bigg|_{t=0} \\
&\qquad = ijx^{j-1}y^{N-j+1} + i(N-j)x^{j+1}y^{N-j-1}.
\end{aligned}$$

In the normalized base, we get

$$\begin{aligned}
\frac{d}{dt}(\exp tX_3)(\widehat{e}_j)\Bigg|_{t=0} &= \frac{1}{\sqrt{j!\,(N-j)!}}\,\frac{d}{dt}(\exp tX_3)(e_j)\Bigg|_{t=0} \\
&= \frac{1}{\sqrt{j!\,(N-j)!}} ijx^{j-1}y^{N-j+1} + \frac{1}{\sqrt{j!\,(N-j)!}} i(N-j)x^{j+1}y^{N-j-1} \\
&= \frac{\sqrt{(j-1)!\,(N-j+1)!}}{\sqrt{j!\,(N-j)!}} \frac{ijx^{j-1}y^{N-j+1}}{\sqrt{(j-1)!\,(N-j+1)!}} \\
&\qquad + \frac{\sqrt{(j+1)!\,(N-j-1)!}}{\sqrt{j!\,(N-j)!}} \frac{i(N-j)x^{j+1}y^{N-j-1}}{\sqrt{(j+1)!\,(N-j-1)!}} \\
&= \sqrt{\frac{(N-j+1)}{j}}\, ij\widehat{e}_{j-1} + \sqrt{\frac{j+1}{(N-j)}}\, i(N-j)\widehat{e}_{j-1} \\
&= i\sqrt{j(N-j+1)}\,\widehat{e}_{j-1} + i\sqrt{(j+1)(N-j)}\,\widehat{e}_{j+1}.
\end{aligned}$$

Lastly, for $j = 0$ and $j = N$, one can find $d\pi_N(X_3)(\widehat{e}_0) = i\sqrt{N}\widehat{e}_1$ and $d\pi_N(X_3)(\widehat{e}_N) = i\sqrt{N}\widehat{e}_{N-1}$, respectively.

We conclude that

$$d\pi_N(X_3)(\widehat{e}_j) = \begin{cases} i\sqrt{N}\widehat{e}_1, & \text{if } j = 0; \\ i\sqrt{j(N-j+1)}\,\widehat{e}_{j-1} + i\sqrt{(j+1)(N-j)}\,\widehat{e}_{j+1}, & \text{if } 0 < j < N; \\ i\sqrt{N}\widehat{e}_{N-1}, & \text{if } j = N. \end{cases}$$

6. Testing uniform distribution: even N

There are several ways of testing whether a sequence of points becomes uniformly distributed on S^N.

One can compute the $l^k, k \geq 2$ norms of eigenvectors, as well as their l^1 norm, and study its behaviour.

One can project eigenvectors onto an m-dimensional subspace of $\mathbb{R}^{N+1}$, and compare the distribution of the projections to the m-dimensional Gaussian.

Now, for N even, the eigenvalues will be simple. Since $\pi_N(g)$ is self-adjoint and as the eigenvalues are simple, it follows that the eigenvectors will be real. Hence, we will plot the increasing l^p norm of the eigenvectors of $\pi_N(g)$ for even p against the eigenvalue number (in absolute value), and the same procedure will be done for the l^∞ norm.

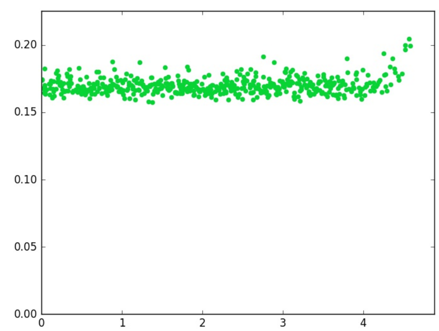


Figure 1. l^6 norm of the eigenvectors of $\pi_N(g)$ for $N = 500$ against eigenvalue number (in absolute value)

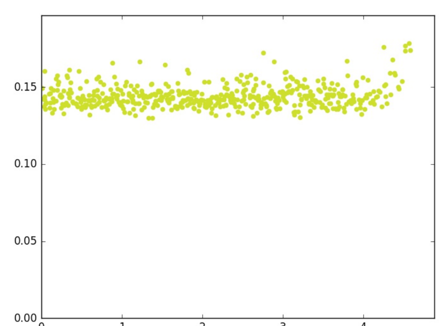


Figure 2. l^8 norm of the eigenvectors of $\pi_N(g)$ for $N = 500$ against eigenvalue number (in absolute value)

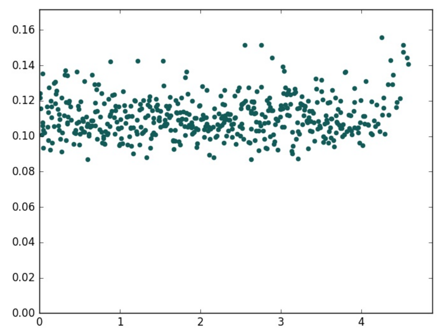


FIGURE 3. l^∞ norm of the eigenvectors of $\pi_N(g)$ for $N = 500$ against eigenvalue number (in absolute value)

The above graphs are to be expected, since l^p is a decreasing function to l^∞ as $p \to \infty$.

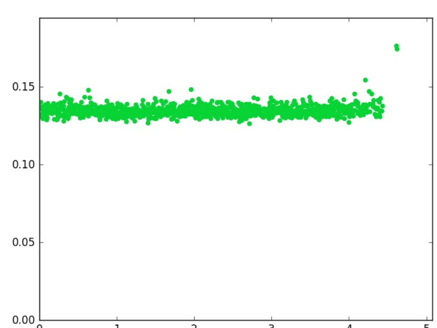


FIGURE 4. l^6 norm of the eigenvectors of $\pi_N(g)$ for $N = 1000$ against eigenvalue number (in absolute value)

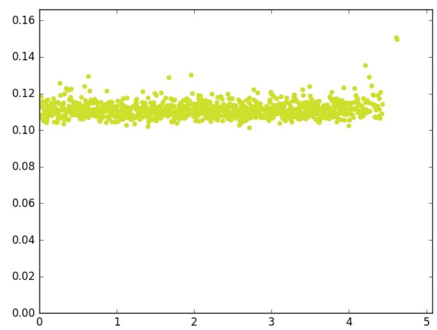


FIGURE 5. l^8 norm of the eigenvectors of $\pi_N(g)$ for $N = 1000$ against eigenvalue number (in absolute value)

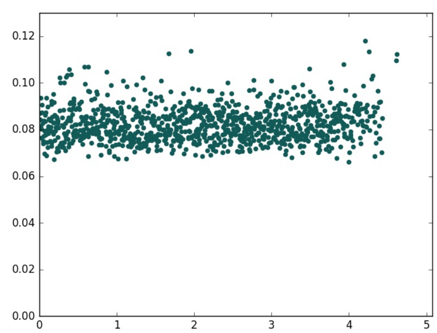


FIGURE 6. l^∞ norm of the eigenvectors of $\pi_N(g)$ for $N = 1000$ against eigenvalue number (in absolute value)

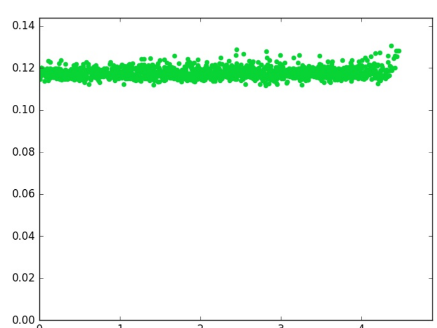


FIGURE 7. l^6 norm of the eigenvectors of $\pi_N(g)$ for $N = 1500$ against eigenvalue number (in absolute value)

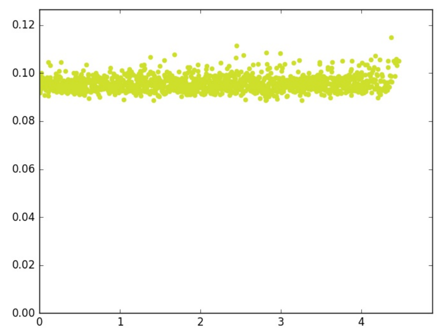


FIGURE 8. l^8 norm of the eigenvectors of $\pi_N(g)$ for $N = 1500$ against eigenvalue number (in absolute value)

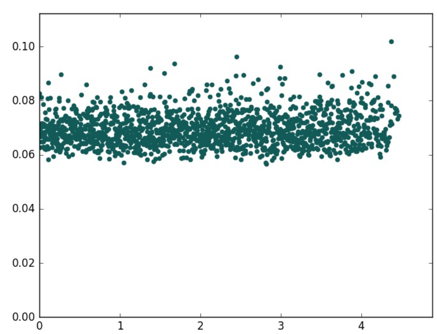
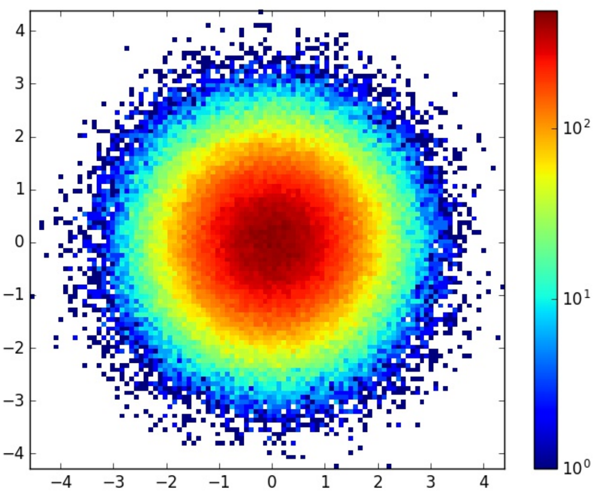


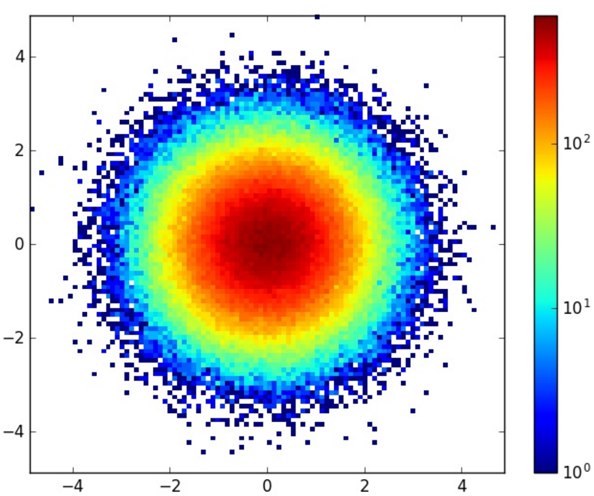
FIGURE 9. l^∞ norm of the eigenvectors of $\pi_N(g)$ for $N = 1500$ against eigenvalue number (in absolute value)

Now, we will make a histogram of the components of the eigenvectors of $\pi_N(g)$ (which are complex) and compare it against a histogram of a random variable distributed according to a bivariate normal distribution. The histogram of the eigenvectors will be made by taking all the eigenvectors and all of their components (making a 2-dimensional histogram of the real and imaginary parts of the components of the eigenvectors). For the other histogram, we will take $(N+1)^2$ samples from a bivariate normal distribution.

Numerically, it was found that the components of the eigenvectors considered as a bivariate normal random variable (as a bivariate (X, Y) random variable, where X is the real part of the components of the eigenvectors and Y the imaginary part) have an identity covariance matrix, so the histogram of the eigenvectors was compared against a bivariate normal distribution with 0 mean and an identity covariance matrix. To eliminate the bias present in the eigenvectors of the matrix, if $u = (u_1, \ldots, u_{N+1})$ is a complex eigenvector, we take $\arg u_1$ and multiply the eigenvector by $e^{-i\arg(u_1)}$. We will multiply the components by $\sqrt{2(N+1)}$ (in order to obtain a normal distribution). In both figures the generators were fixed and N increased.

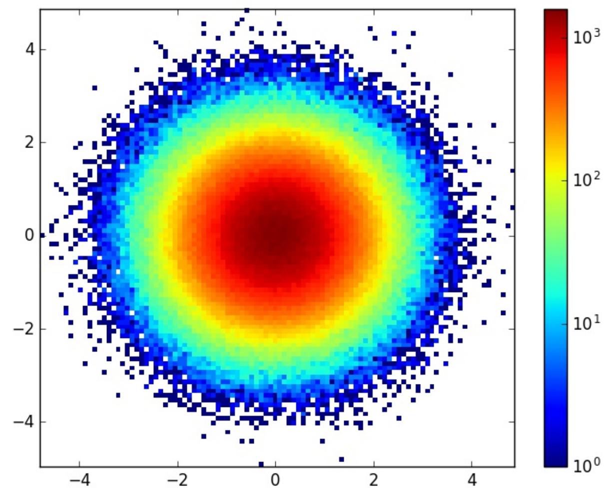


(a) Histogram of the bivariate gaussian

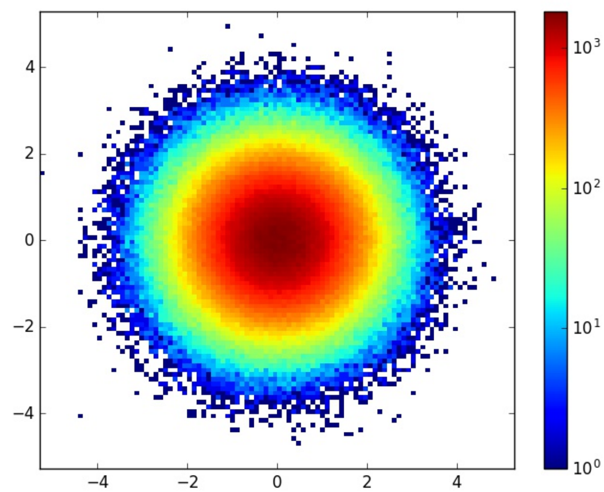


(b) Histogram of the eigenvector components

FIGURE 10. Histogram of the bivariate gaussian distribution against the histogram of the eigenvector components of $\pi_N(g)$ for $N = 600$



(a) Histogram of the bivariate gaussian



(b) Histogram of the eigenvector components

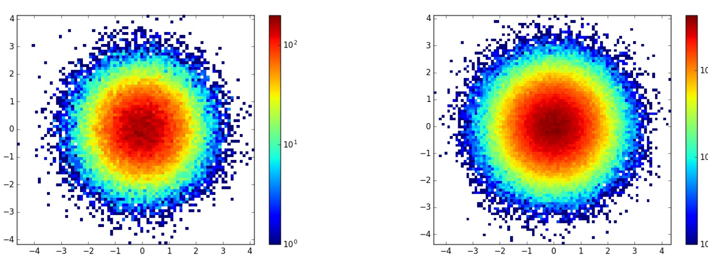
FIGURE 11. Histogram of the bivariate gaussian distribution against the histogram of the eigenvector components of $\pi_N(g)$ for $N = 1000$

7. Testing uniform distribution: odd N

For odd N, the eigenvalues will be double ($\pi_N(g)$ has even dimension). In a similar way to the histograms produced for the even case, we do the same normalization process for the eigenvectors. Now, for each eigenvalue λ_i, we will have two associated eigenvectors (which form a basis for the eigenspace associated to λ_i) u_i and u_{i+1}. Then, consider the random variable (X, Y, Z, W), where we have:

$$X = \mathrm{Re}(u_{i_j}) \quad Y = \mathrm{Im}(u_{i_j}) \quad Z = \mathrm{Re}(u_{i+1_j}) \quad X = \mathrm{Im}(u_{i+1_j})$$

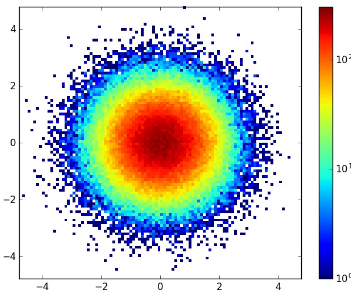
For u_{i_j} being the j-th component of u_i. Numerically, the random variable (X, Y, Z, W) seems to have 0 mean and an identity covariance matrix as N becomes large. We consider the random variables coming from projections of (X, Y, Z, W), $(X, Y), (X, W), (Z, Y), (Z, W)$ and fixed generators for $\pi_N(g)$. The histogram for the projected random variables will be compared against a histogram of $\frac{(N+1)^2}{4}$ samples from a bivariate standard normal distribution (the projection random variables had as well 0 mean and an identity covariance matrix).



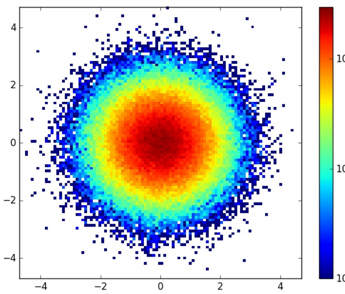
(a) Histogram of the bivariate gaussian for 601

(b) Histogram of the bivariate gaussian for $N = 1001$

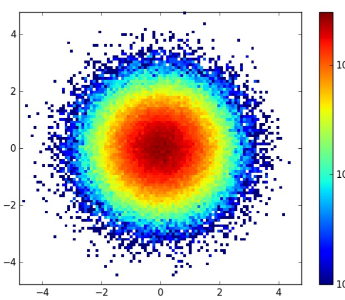
Figure 12. Histogram of the bivariate standard gaussian distribution for samples corresponding to $N = 601$ and $N = 1001$



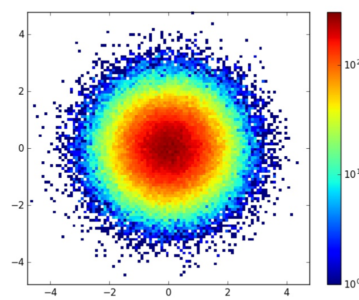
(a) Histogram of (X, Y)



(b) Histogram of (X, W)

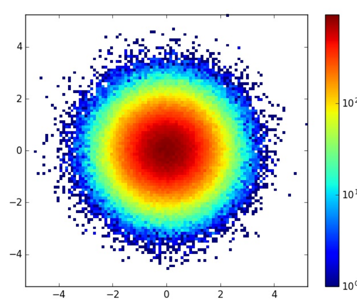


(c) Histogram of (Z, Y)

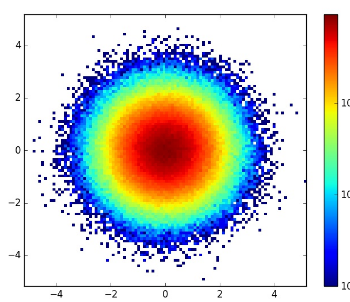


(d) Histogram of (Z, W)

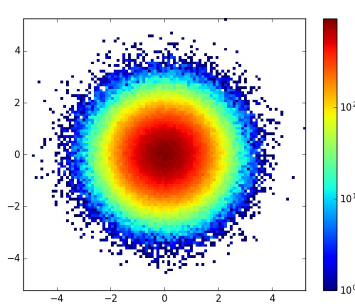
FIGURE 13. Histograms of projections of (X, Y, Z, W) for $N = 601$



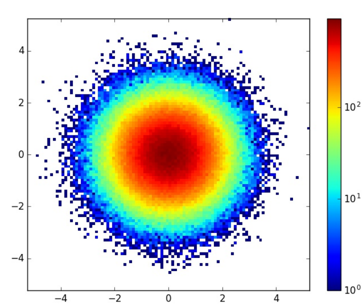
(a) Histogram of (X, Y)



(b) Histogram of (X, W)



(c) Histogram of (Z, Y)



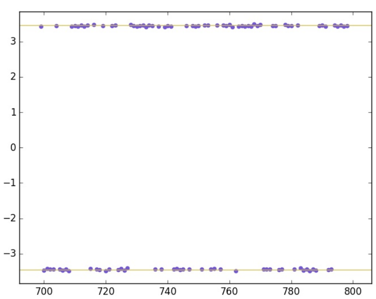
(d) Histogram of (Z, W)

FIGURE 14. Histograms of projections of (X, Y, Z, W) for $N = 1001$

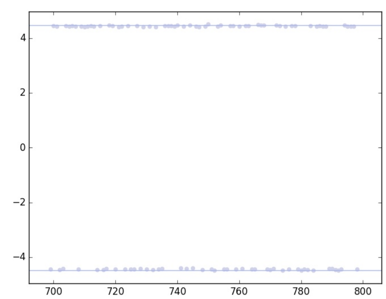
8. Ramanujan Bounds

Another interesting point is to analyze the limiting behavior of the eigenvectors of $\pi_N(g)$ for different number of generators (which are randomly chosen). It is expected that the largest eigenvalues (in absolute value) of $\pi_N(g)$ for any number k of generators tend to $\pm 2\sqrt{2k-1}$ as $n \to \infty$, also known as the Ramanujan bound. This assertion has an equivalent for random graphs, also known as Alon's conjucture, proven recently.

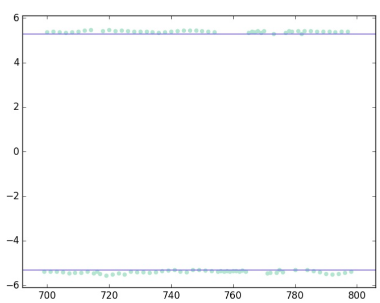
The statement will be verified numerically for N in a range between 699 and 799, for a varying number of generators (specifically, for 2, 3, 4 and 5 generators). The largest eigenvalue number will be compared against N, where the straight lines in the graphs represent the Ramanujan bound.



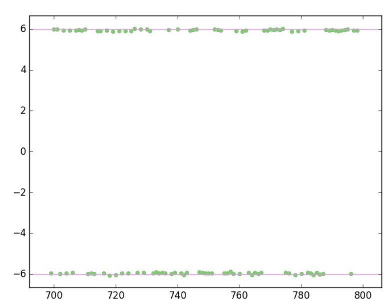
(a) Largest eigenvalues for $k = 2$



(b) Largest eigenvalues for $k = 3$



(c) Largest eigenvalues for $k = 4$



(d) Largest eiegenvalues for $k = 5$

Figure 15. Largest eigenvalue number of $\pi_N(g)$ for $k = 2, 3, 4, 5$ generators chosen at random

9. Representation varieties

Moving in the space of representations. LATER ON.

10. The Spectral Function

It is also interesting to look at the behaviour of the spectral function

$$F(i,j,\mu) = \sum_{\lambda_m \le \mu} u_m(i)\overline{u_m(j)}, \tag{10.1}$$

where λ_m, $1 \le m \le N+1$, are the eigenvalues of $\pi_N(g)$ and $u_m(i)$,$u_m(j)$ are the i-th and j-th component, respectively, of the corresponding eigenvector. Note that the spectral function defined in (10.1) is "self-normalizing" (as a kernel for a spectral projection operator), in the sense of Definition (i) from [TV].

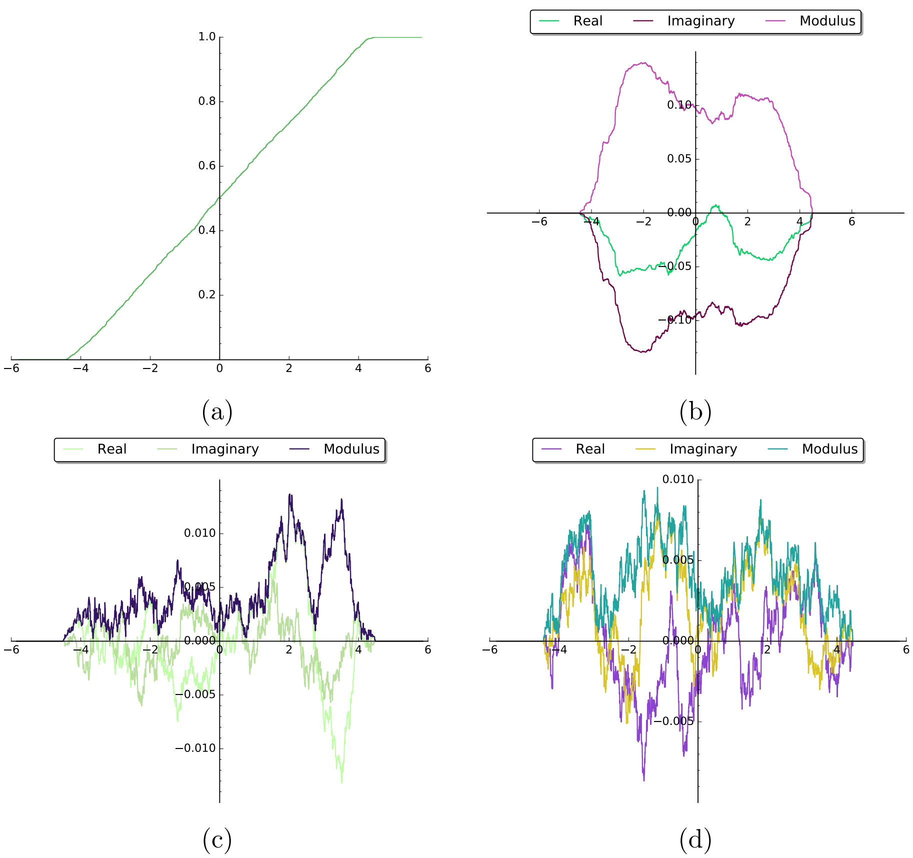


Figure 17. Shown above are plots of the spectral function, $F(i,j,\mu)$ of $\pi_N(g)$, $N = 2000$, $g \in \mathbb{R}[\mathrm{SU}(2)]$ fixed, for various i and j: (a) $F(18,18,\mu)$ (b) $F(0,1,\mu)$ (c) $F(13,23,\mu)$(d) $F(13,113,\mu)$

Thing to do: look at matrix entries of $\pi_N(g)$, and see if one can notice anything interesting.

One way to gain qualitative information about $\pi_N(g)$ is to make a color map of the matrix. Considering a real matrix $A_N(g)$, whose entries are related to the entries of $\pi_N(g)$ by some scalar function (modulus, real part etc.), we can color an $N \times N$ grid according to the entries of $A_N(g)$ and some color scale.

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

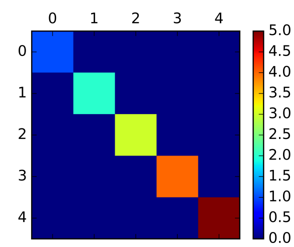


FIGURE 18. Example of a color map of the matrix B shown above.

We can apply this to a matrix $A_N(g)$, where we take the entries of $A_N(g)$ to be the modulus of the entries of $\pi_N(g)$. Doing this gives some interesting results.

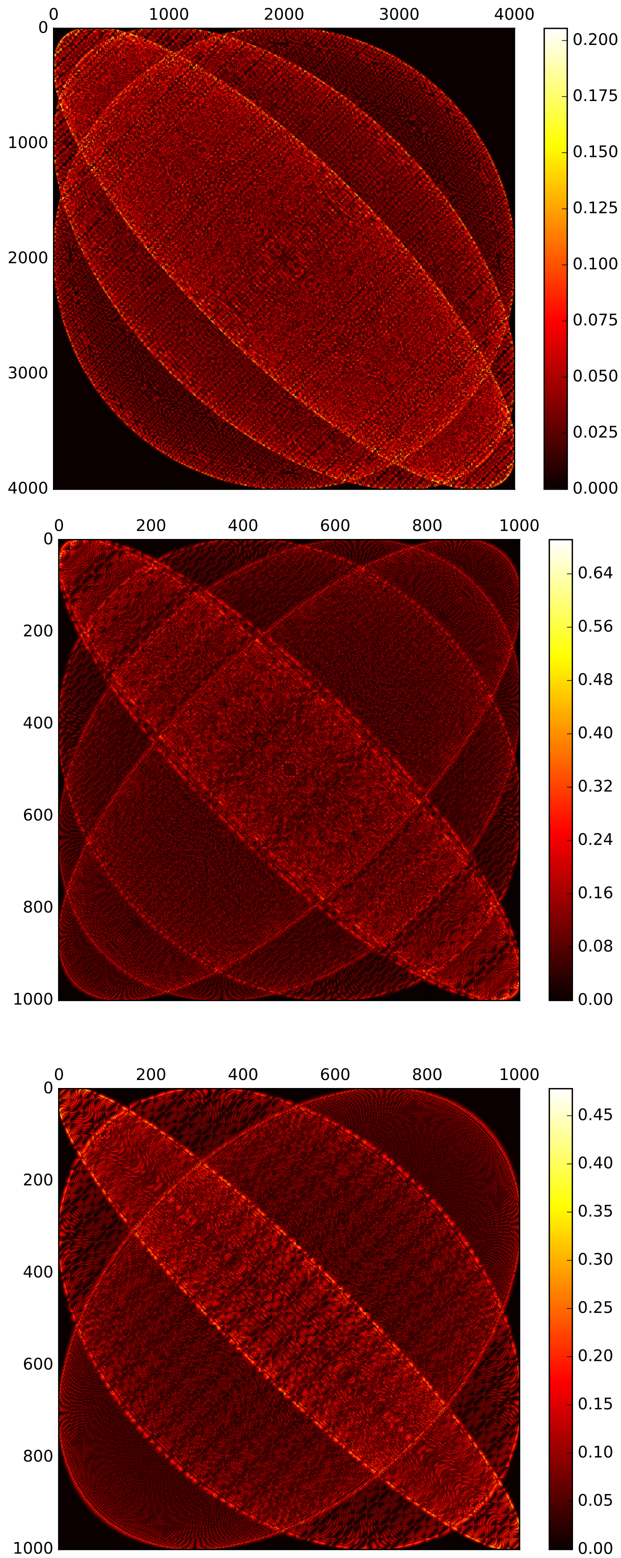


FIGURE 19. Color maps of the matrix $A_N(g)$ for various $g \in \mathbb{R}[\mathrm{SU}(2)]$ and $N \in \mathbb{N}$.

While different $g \in \mathbb{R}[\mathrm{SU}(2)]$ produce different maps, they all seem to share the similarity of having a family of ellipses inscribed within the matrix grid.

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